1.

a) According to Gauss's law, the electric flux Φ will be

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_V \rho dV = \frac{1}{\epsilon_0} \rho \times V = \frac{1}{\epsilon_0} \rho \times L^3 = \frac{16}{\epsilon_0} \times 10^{-9}\,\mathrm{Vm},$$

b) If $\rho(x, y, z) = x^2 + y^2 + z^2 \,\frac{\mathrm{C}}{\mathrm{m}^3}$ (within the cube), the total electric flux can be computed as follows:

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_V x^2 + y^2 + z^2 \, dV$$

$$= \frac{1}{\epsilon_0} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} x^2 + y^2 + z^2 \, dx \, dy \, dz = \frac{8}{\epsilon_0} \times 10^{-15}\,\mathrm{Vm}.$$

c) If the medium is homogeneous, then the displacement flux is

$$\begin{aligned} \oint_S \mathbf{D} \cdot d\mathbf{S} &= \epsilon_0 \oint_S \mathbf{E} \cdot d\mathbf{S} \\ &= 8 \times 10^{-15}\,\mathrm{C}. \end{aligned}$$

d) Since the charge distribution is symmetrical around the center, the flux for any of the square surfaces is one-sixth of the total value, i.e., $\frac{4}{3} \times 10^{-15}\,\mathrm{C}$.

2. Two unknown charges, Q_1 and Q_2 are located at $(1, 0, 0)$ and $(-1, 0, 0)$, respectively. The displacement flux $\oint_S \mathbf{D} \cdot d\mathbf{S}$ through the $yz-$plane in the $\hat{x}$ direction is $-2\,\text{C}$, thus,

$$-\frac{Q_1}{2} + \frac{Q_2}{2} = -2\,C \tag{1}$$

Also, the displacement flux through the plane $y = 1$ in the $-\hat{y}$ direction is $1\,\text{C}$, which implies that

$$-\frac{Q_1}{2} - \frac{Q_2}{2} = 1\,C \tag{2}$$

Using these two equations, we get

$$Q_1 = 1\,\text{C} \qquad \text{and} \qquad Q_2 = -3\,\text{C}.$$

3.

(a) Given that $\rho_l = 2\ C/m,\ r = 1\ m,\ L = 2\ m$, we can apply cylindrical symmetry to set up the surface integral while ignoring the end caps ($\mathbf{D} \perp$ endcaps).

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho dV$$

$$\int_V \rho dV = Q_V = \rho_l L$$

Therefore,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \rho_l L = 2\frac{C}{m} \cdot 2\ m = 4C$$

(b) Yes, the flux integral would change. While $Q_V = \rho_l L$ is valid for both infinite and truncated lines of charge, the $\mathbf{E}$-field due to an infinite line of charge is strictly radial, whereas a truncated line of charge will have "fringing" fields that cross through the end caps of the cylindrical surface. Therefore, we cannot treat the end caps as having zero contribution to the total displacement flux. Instead, the flux due to fringing fields will cause the flux through the curved part of the cylinder to decrease.

4.

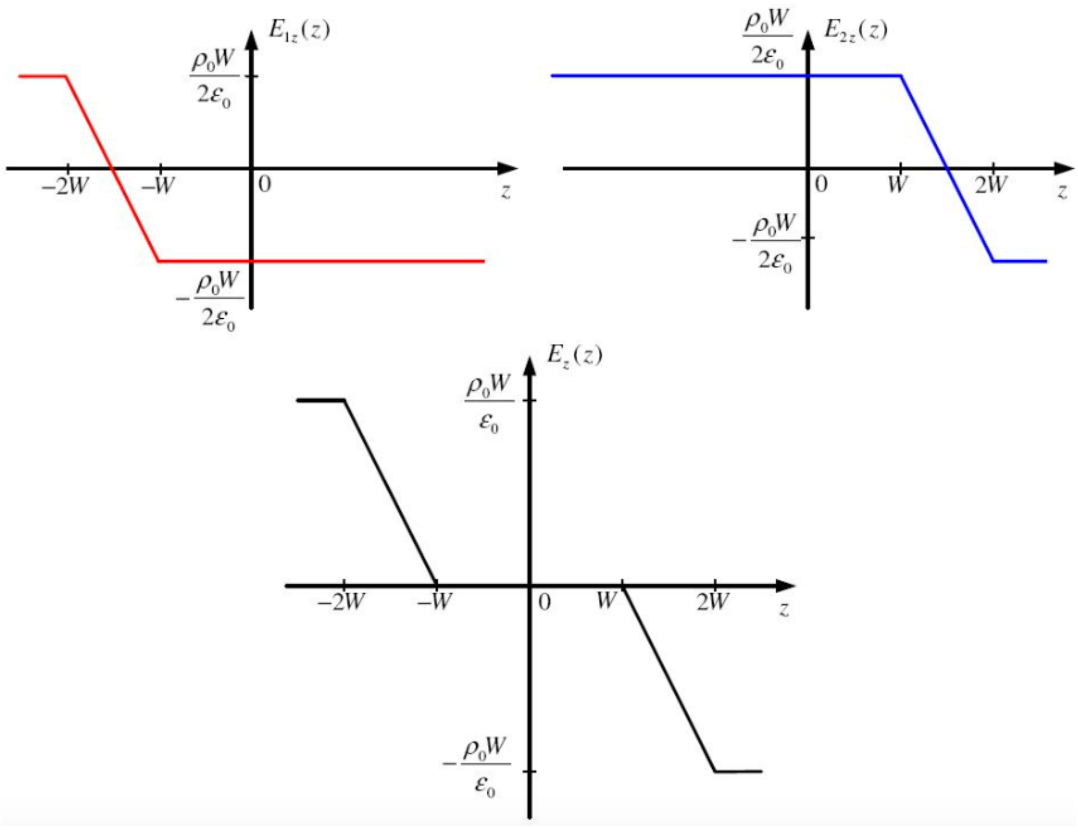
a) A two-slab geometry of two identical slabs of equal widths W in z−direction and expanding infinitely in x and y directions is considered here. The electric field generated by a single charged slab has been derived in the class notes. Making use of that result, we find that the electric fields generated by each of the two slabs are:

$$\mathbf{E}_1 = \begin{cases} \frac{\rho_0 W}{2\epsilon_0}\,\hat{z}, & z < -2W, \\ -\frac{\rho_0}{\epsilon_0}(z + \frac{3W}{2})\,\hat{z}, & -2W < z < -W, \\ -\frac{\rho_0 W}{2\epsilon_0}\,\hat{z}, & z > -W \end{cases}$$

$$\mathbf{E}_2 = \begin{cases} -\frac{\rho_0 W}{2\epsilon_0}\,\hat{z}, & z > 2W, \\ -\frac{\rho_0}{\epsilon_0}(z - \frac{3W}{2})\,\hat{z}, & W < z < 2W, \\ \frac{\rho_0 W}{2\epsilon_0}\,\hat{z}, & z < W \end{cases}$$

respectively. Adding these fields, we find that the total field in space

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \begin{cases} \frac{\rho_0 W}{\epsilon_0}\,\hat{z}, & z < -2W \\ -\frac{\rho_0}{\epsilon_0}(z + W)\,\hat{z}, & -2W < z < -W \\ 0, & -W < z < W \\ -\frac{\rho_0}{\epsilon_0}(z - W)\,\hat{z}, & W < z < 2W \\ -\frac{\rho_0 W}{\epsilon_0}\,\hat{z}, & z > 2W \end{cases}$$



b) From $\mathbf{D} = \epsilon_0 \mathbf{E}$,

$$\nabla \cdot \mathbf{D} = \begin{cases} 0, & z < -2W \\ -\rho_0, & -2W < z < -W \\ 0, & -W < z < W \\ -\rho_0 & W < z < 2W \\ 0 & z > 2W \end{cases}$$

5.

a) For $\rho_{s_1} = 8\,\frac{\text{C}}{\text{m}^2}$, the displacement vectors at the origin due to charge sheets 1 and 2 can be written as

$$\begin{aligned}
\mathbf{D}_1 &= \frac{1}{2}\rho_{s_1}\hat{x} = 4\hat{x}\,\frac{\text{C}}{\text{m}^2}, \text{ and} \\
\mathbf{D}_2 &= \frac{1}{2}\rho_{s_2}(-\hat{x}) = -\frac{1}{2}\rho_{s_2}\hat{x},
\end{aligned}$$

respectively. The resultant displacement vector is given by $\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2 = (4 - \frac{1}{2}\rho_{s_2})\hat{x} = 6\,\frac{\text{C}}{\text{m}^2}$. Therefore, we can determine that the charge density ρ_{s_2} is $-4\,\frac{\text{C}}{\text{m}^2}$.

b) Following the same procedure, the displacement vectors at the origin due to charge sheets 1 and 2 can be written as

$$\begin{aligned}
\mathbf{D}_1 &= \frac{1}{2}\rho_{s_1}\hat{x}, \text{ and} \\
\mathbf{D}_2 &= \frac{1}{2}\rho_{s_2}(-\hat{x}) = -\frac{1}{2}\rho_{s_2}\hat{x},
\end{aligned}$$

respectively. Given that $\rho_{s_1} = -\rho_{s_2}$, the resultant displacement vector is given by $\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2 = -\rho_{s_2}\hat{x} = 6\hat{x}\,\frac{\text{C}}{\text{m}^2}$. As a result, the charge density ρ_{s_2} is determined to be $-6\,\frac{\text{C}}{\text{m}^2}$.

6.

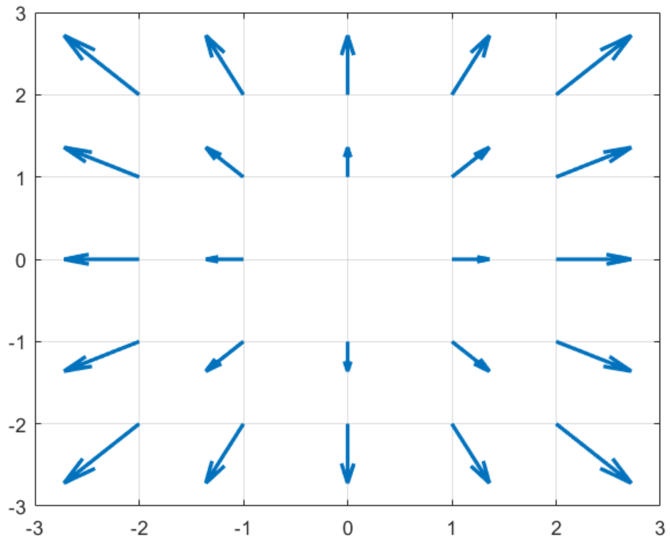
a) Curl and divergence of the vector field $\mathbf{F} = x\hat{x} + y\hat{y}$ are

$$\nabla \times \mathbf{F} = \nabla\times(x\hat{x} + y\hat{y}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = 0,$$

and

$$\nabla \cdot \mathbf{F} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (x\hat{x} + y\hat{y}) = 2,$$

respectively. When we sketch the vector field $\mathbf{F}$, we obtain the following figure



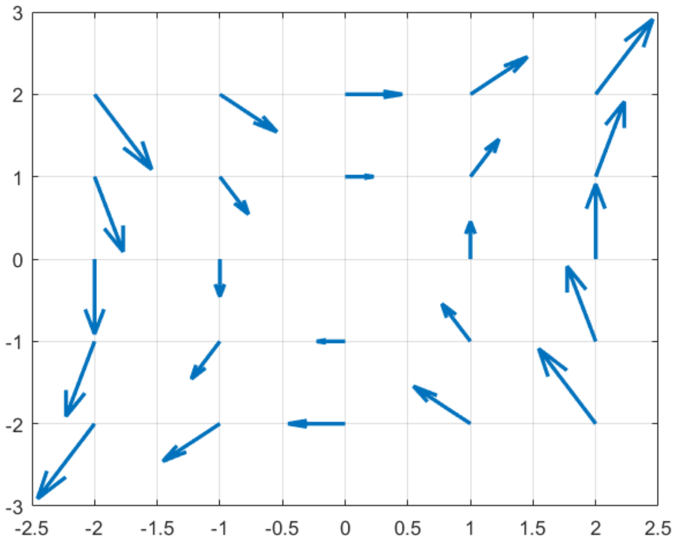
b) Curl and divergence of the vector field $\mathbf{F} = y\hat{x} + 2x\hat{y}$ are

$$\nabla \times \mathbf{F} = \nabla \times (y\hat{x} + 2x\hat{y}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 2x & 0 \end{vmatrix} = \hat{z},$$

and

$$\nabla \cdot \mathbf{F} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (y\hat{x} + 2x\hat{y}) = 0,$$

respectively. When we sketch the vector field $\mathbf{F}$, we obtain the following figure



c) **Curl** is a vector operator that describes the rotation of a 3-D vector field. In short, it is the vector representing circulation per unit area. Since the vector field $\mathbf{F}$ in part (b) is defined only by the curl operator (i.e. divergence-free), $\nabla \times \mathbf{F} \neq 0$ implies the field strength varies **across** the direction of the field.

d) **Divergence** is an operator picking up the variation of the field strength along the direction of the vector field. As seen in part (a), the divergence operator picks up the field strength of the vector $\mathbf{F} = x\hat{x} + y\hat{y}$ which is curl-free. Therefore, $\nabla \cdot \mathbf{F} \neq 0$ implies the field strength varies **along** the direction of the field.