1. Consider the 3-D vectors

$$\begin{aligned}\mathbf{A} &= \hat{x} + 3\hat{y} - 2\hat{z},\\ \mathbf{B} &= \hat{x} + \hat{y} - \hat{z},\\ \mathbf{C} &= -\hat{x} + 3\hat{y} + 2\hat{z},\end{aligned}$$

a) The vector

$$\mathbf{A} - \mathbf{B} + 3\mathbf{C} = -3\hat{x} + 11\hat{y} + 5\hat{z}$$

b) The vector *magnitude*

$$|\mathbf{A} - \mathbf{B} + 3\mathbf{C}| = \sqrt{3^2 + 11^2 + 5^2} = \sqrt{155} \approx 12.45$$

c) The unit vector

$$\hat{u} = \frac{\mathbf{A} + 2\mathbf{B} - \mathbf{C}}{|\mathbf{A} + 2\mathbf{B} - \mathbf{C}|} = \frac{4\hat{x} + 2\hat{y} - 6\hat{z}}{\sqrt{4^2 + (2)^2 + (-6)^2}} \approx 0.53\hat{x} + 0.27\hat{y} - 0.80\hat{z}$$

d) The *dot product*

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} = (\hat{x} + 3\hat{y} - 2\hat{z}) \cdot (\hat{x} + \hat{y} - \hat{z}) = 1 \times 1 + 3 \times 1 + (-2) \times (-1) = 6$$

e) The angle θ between **A** and **B** can be found by using properties of the dot product.

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$$

$$|\mathbf{A}| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$$

$$|\mathbf{B}| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$$

$$\mathbf{A} \cdot \mathbf{B} = (1)(1) \; + (3)(1) \; + (-2)(-1) = 6$$

$$\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{6}{\sqrt{14}\sqrt{3}} = \cos\theta$$

$$\theta = \cos^{-1}(\frac{6}{\sqrt{14}\sqrt{3}}) = 22.21^\circ$$

f) The *cross product*

$$\begin{aligned}\mathbf{B} \times \mathbf{C} = -\mathbf{C} \times \mathbf{B} &= (\hat{x} + \hat{y} - \hat{z}) \times (-\hat{x} + 3\hat{y} + 2\hat{z})\\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & -1 \\ -1 & 3 & 2 \end{vmatrix}\\ &= 5\hat{x} - \hat{y} + 4\hat{z}\end{aligned}$$

g) Since $\mathbf{B} \cdot \mathbf{C} = -1 + 3 - 2 = 0$, **B** and **C** are orthogonal.

2. We know$Q_1 = -1C$, and $Q_2 = +1C$. The electric field at the point P_3 is the superposition of those induced by Q_1 and Q_2, namely

$$\begin{aligned}
\mathbf{E_3} &= \sum_{i=1}^{2} \frac{Q_i}{4\pi\varepsilon_0 |\mathbf{r_3} - \mathbf{r_i}|^2} \cdot \frac{\mathbf{r_3} - \mathbf{r_i}}{|\mathbf{r}_3 - \mathbf{r}_i|} \\
&= (-1)\,\frac{\hat{x}}{4\pi\varepsilon_0 |\hat{x}|^3} + (1)\,\frac{-\hat{x}}{4\pi\varepsilon_0 |-\hat{x}|^3} \\
&= \frac{-2}{4\pi\varepsilon_0}\hat{x}\,(V/m). \\
&\approx -1.798 \times 10^{10}\hat{x}\,(V/m).
\end{aligned}$$

The electric field at the point P_4 and P_5 can be obtained in a similar way

$$\begin{aligned}
\mathbf{E_4} &= \sum_{i=1}^{2} \frac{Q_i}{4\pi\varepsilon_0 |\mathbf{r_4} - \mathbf{r_i}|^2} \cdot \frac{\mathbf{r_4} - \mathbf{r_i}}{|\mathbf{r}_4 - \mathbf{r}_i|} \\
&= (-3)\,\frac{\hat{x}}{4\pi\varepsilon_0 |3\hat{x}|^3} + (1)\,\frac{\hat{x}}{4\pi\varepsilon_0 |\hat{x}|^3} \\
&= \frac{2}{9\pi\varepsilon_0}\hat{x}\,(V/m). \\
&\approx 7.989 \times 10^{9}\hat{x}\,(V/m).
\end{aligned}$$

$$\begin{aligned}
\mathbf{E_5} &= \sum_{i=1}^{2} \frac{Q_i}{4\pi\varepsilon_0 |\mathbf{r_5} - \mathbf{r_i}|^2} \cdot \frac{\mathbf{r_5} - \mathbf{r_i}}{|\mathbf{r}_5 - \mathbf{r}_i|} \\
&= (-1)\,\frac{-\hat{z} + \hat{x}}{4\pi\varepsilon_0 |-\hat{z} + \hat{x}|^3} + (1)\,\frac{-\hat{z} - \hat{x}}{4\pi\varepsilon_0 |-\hat{z} - \hat{x}|^3} \\
&= \frac{-1}{4\pi\varepsilon_0 \sqrt{2}}\hat{x}\,(V/m). \\
&\approx -6.355 \times 10^{9}\hat{x}\,(V/m).
\end{aligned}$$

To find the force $\mathbf{F}$ experienced by a charge $Q = 2C$ at the origin, simply evaluate $\mathbf{F} = Q\mathbf{E_3}$, since $\mathbf{E_3}$ is the static field at the origin generated by Q_1 and Q_2.

$$\mathbf{F} = Q\mathbf{E_3}$$

$$\mathbf{F} = 2C \;\cdot -1.798 \times 10^{10}\hat{x}\,(V/m) \approx -3.5950 \times 10^{10}\ N.$$

3. In the three cases, we have the same **E** and **B** at the origin, but different **v**. Thus, we have different **F**. In each case, **v** and **F** must satisfy $\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$. Therefore, we have the following three equations

$$\begin{cases} & & 2\hat{x} - \hat{y} & = & \mathbf{E} \\ \mathbf{F_2} - \mathbf{E} & = & -2\hat{y} & = & \mathbf{v_2} \times \mathbf{B} \\ \mathbf{F_3} - \mathbf{E} & = & \hat{x} - \hat{y} - \hat{z} & = & \mathbf{v_3} \times \mathbf{B} \end{cases}$$

Assume that $\mathbf{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$, we obtain

$$\begin{cases} \mathbf{E} & = & 2\hat{x} - \hat{y} \\ B_y & = & 0 \\ -2B_x\hat{y} & = & -2\hat{y} \\ (\hat{x} - \hat{y} - \hat{z}) \cdot \mathbf{B} & = & 0 \end{cases}$$

Then

$$\begin{cases} B_y & = & 0 \\ B_x & = & 1 \\ B_x - B_z & = & 0 \end{cases}$$

Thus

$$\begin{cases} B_y & = & 0 \\ B_x = B_z & = & 1 \end{cases}$$

In summary, $\mathbf{E} = 2\hat{x} - \hat{y}\,(V/m)$ and $\mathbf{B} = \hat{x} + \hat{z}\,(Wb/m^2)$.

4. Consider a scalar field

$$V = x^2 + y^2 + z^2$$

a) To find the gradient ∇V,

$$\nabla V = \hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z}$$
$$= 2x\hat{x} + 2y\hat{y} + 2z\hat{z}$$

b) To find ∇V at (1,2,3),

$$\nabla V(1,2,3) = 2(1)\hat{x} + 2(2)\hat{y} + 2(3)\hat{z}$$
$$= 2\hat{x} + 4\hat{y} + 6\hat{z}$$

c) The gradient of a function is a vector field in which every vector points in the direction of the rate of greatest increase. In our case, ∇V points in the direction where V has the greatest rate of increase at (1,2,3).

5. Consider a vector field

$$\mathbf{E} = \hat{x}xy + \hat{y}yz + \hat{z}zx$$

a) To find the divergence, $\nabla \cdot \mathbf{E}$,

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\nabla \cdot \mathbf{E} = y + z + x = y + z + x$$

b) $\nabla \cdot \mathbf{E}$ at (1,1,1) = 3 by inspection of the result in part a).

c) To find the curl, $\nabla \times \mathbf{E}$,

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \hat{x}(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}) + \hat{y}(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}) + \hat{z}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y})$$

$$= \hat{x}(0 - y) + \hat{y}(0 - z) + \hat{z}(0 - x) = -y\hat{x} - z\hat{y} - x\hat{z}$$

d) By definition, $\nabla \cdot (\nabla \times \mathbf{E}) = 0$. Using the definitions of divergence and curl,

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\nabla \times \mathbf{E} = \hat{x}(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}) + \hat{y}(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}) + \hat{z}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y})$$

$$\nabla \cdot (\nabla \times \mathbf{E}) = \frac{\partial}{\partial x}(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y})$$

$$\nabla \cdot (\nabla \times \mathbf{E}) = \frac{\partial^2 E_z}{\partial x \partial y} - \frac{\partial^2 E_y}{\partial x \partial z} + \frac{\partial^2 E_x}{\partial y \partial z} - \frac{\partial^2 E_z}{\partial y \partial x} + \frac{\partial^2 E_y}{\partial z \partial x} - \frac{\partial^2 E_x}{\partial z \partial y}$$

Since $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$ by Clairaut's Theorem, $\nabla \cdot (\nabla \times \mathbf{E}) = 0$.

e) To find $\int_{(0,0,0)}^{(1,1,1)} \mathbf{E} \cdot d\mathbf{l}$ along the path $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$, let the path $(0,0,0) \to (0,0,1)$ be C_1, $(0,0,1) \to (0,1,1)$ be C_2, and $(0,1,1) \to (1,1,1)$ be C_3.

$$\int_{(0,0,0)}^{(1,1,1)} \mathbf{E} \cdot d\mathbf{l} = \int_{C_1} \mathbf{E} \cdot d\mathbf{l} \; + \int_{C_2} \mathbf{E} \cdot d\mathbf{l} \; + \int_{C_3} \mathbf{E} \cdot d\mathbf{l}$$

$$\begin{aligned}
\int_{(0,0,0)}^{(1,1,1)} \mathbf{E} \cdot d\mathbf{l} &= \int_{(0,0,0)}^{(0,0,1)} \mathbf{E} \cdot \hat{z} dz \; + \int_{(0,0,1)}^{(0,1,1)} \mathbf{E} \cdot \hat{y} dy \; + \int_{(0,1,1)}^{(1,1,1)} \mathbf{E} \cdot \hat{x} dx \\
&= \int_0^1 z \cdot 0 dz \; + \int_0^1 y \cdot 1 dy \; + \int_0^1 x \cdot 1 dx \\
&= 0 + (\frac{y^2}{2})\Big|_0^1 + (\frac{x^2}{2})\Big|_0^1 = 1
\end{aligned}$$