1.

a) As the wave travels at the speed of light

$$\mu = \mu_o$$

and the impedance is given by

$$\eta = \eta_o$$

b) Wavelength can be calculated by

$$\lambda = \frac{v}{f} = 0.1\, m$$

Since the wave travels at the speed of light, the above wavelength represents the free space wavelength. This wave lies in the radio wave region and is not visible. Consequently, the wavevector is

$$\beta = \frac{2\pi}{\lambda} = 20\pi\, m^{-1}$$

c) Since the electric field is right hand circular polarized, the form is different from the last two parts,

$$\begin{aligned}
\mathbf{E} &= E_o[\cos(\omega t - \beta z)\hat{x} + \sin(\omega t - \beta z)\hat{y}] \\
&= E_o[\cos(6\pi \times 10^9 t - 20\pi z)\hat{x} + \sin(6\pi \times 10^9 t - 20\pi z)\hat{y}]\, \tfrac{V}{m} \\
\mathbf{H} &= \tfrac{E_o}{\eta}[\cos(\omega t - \beta z)\hat{y} - \sin(\omega t - \beta z)\hat{x}] \\
&= \tfrac{E_o}{120\pi}[\cos(6\pi \times 10^9 t - 20\pi z)\hat{y} - \sin(6\pi \times 10^9 t - 20\pi z)\hat{x}]\, \tfrac{A}{m}
\end{aligned}$$

d) The phasor forms of electric field and magnetic field

$$\tilde{\mathbf{E}} = E_o \exp(-j\beta z)(\hat{x} - j\hat{y}) = E_o \exp(-j20\pi z)(\hat{x} - j\hat{y})\, \tfrac{V}{m}$$
$$\tilde{\mathbf{H}} = \tfrac{E_o}{\eta} \exp(-j\beta z)(\hat{y} + j\hat{x}) = \tfrac{E_o}{120\pi} \exp(-j20\pi z)(\hat{y} + j\hat{x})\, \tfrac{A}{m}$$

e) The complex poynting vector is given by

$$\tilde{\mathbf{S}} = \tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* = \frac{2E_o^2}{\eta}\hat{z} = \frac{E_o^2}{60\pi}\hat{z}$$

and the time average poynting vector by

$$\left\langle \tilde{\mathbf{S}} \right\rangle = \frac{1}{2} Re\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\} = \frac{E_o^2}{\eta}\hat{z} = \frac{E_o^2}{120\pi}\hat{z}$$

2. .

a) Since $\frac{\sigma}{\omega\epsilon} = \frac{4}{2\pi\times10^3\times81\times8.85\times10^{-12}} = 8.88 \times 10^5 \gg 1$, we use the approximations for good conductors, i.e. $\alpha = \beta =\sim \sqrt{\pi f \mu \sigma}$. Therefore, we find

$$\alpha = \beta \approx \sqrt{\pi \times 10^3 \times 4\pi \times 10^{-7} \times 4} = 0.126\,\mathrm{m}^{-1}.$$

Therefore the propagation constant is given by

$$\gamma = \alpha + j\beta \approx 0.126\,(1 + j)\ \mathrm{m}^{-1}.$$

The wavelength is

$$\lambda = \frac{2\pi}{\beta} = 49.87\,\mathrm{m},$$

whereas the penetration depth in the media with the intrinsic impedance

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} e^{j\pi/4} = \sqrt{\frac{2\pi \times 10^3 \times 4\pi \times 10^{-7}}{4}} \left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) = 0.03\,(1+j)\ \Omega,$$

is calculated as

$$\delta = \frac{1}{\alpha} = 7.94\,\mathrm{m}.$$

b) Since $\frac{\sigma}{\omega\epsilon} = \frac{4}{2\pi \times 10^9 \times 81 \times 8.85 \times 10^{-12}} = 0.88$, we will not be able to use the approximations listed in the first table of Lecture 23. Then, The propagation constant is given by $\gamma = \sqrt{j\omega\mu\,(\sigma + j\omega\epsilon)}$ which yields

$$\begin{aligned}\gamma &= \sqrt{(j2\pi \times 10^9 \times 4\pi \times 10^{-7})(4 + j2\pi \times 10^9 \times 81 \times 8.85 \times 10^{-12})} \\ &= 77.46 + j203.87\,\mathrm{m}^{-1}.\end{aligned}$$

Since $\alpha = 77.46\,\mathrm{m}^{-1}$ and $\beta = 203.87\,\mathrm{m}^{-1}$, the penetration depth and the wavelength are

$$\delta = \frac{1}{\alpha} = 0.013\,\mathrm{m} \quad \text{and} \quad \lambda = \frac{2\pi}{\beta} = 0.031\,\mathrm{m}.$$

Finally, the intrinsic impedance is

$$\begin{aligned}\eta &= \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \\ &= \sqrt{\frac{j2\pi \times 10^9 \times 4\pi \times 10^{-7}}{4 + j2\pi \times 10^9 \times 81 \times 8.85 \times 10^{-12}}} \\ &= 33.84 + j12.86\Omega\end{aligned}$$

c) In the ocean, $\sigma = 4\,\mathrm{S/m}$, $\epsilon_r = 81$, and $\mu_r = 1$. For $\omega = 40\pi \times 10^3\,\mathrm{rad/s}$, the propagation constant is

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \sqrt{j0.316 \times (4 + j1.8 \times 10^{-4})} = 0.56 + j0.56\,\mathrm{m}^{-1}.$$

Since $\alpha = 0.56\,\mathrm{m}^{-1}$, the distance at which a submarine should be located in order to receive at least 0.1% of the amplitude of an EM signal transmitted from a ship located at the surface can be calculated as follows

$$e^{-\alpha z} = 0.001 \quad \rightarrow \quad z = -\frac{1}{\alpha}\ln(0.001) = 12.34\,\mathrm{m}.$$

3.

a) For the wave described by $\mathbf{E}_1 = 4\cos(\omega t - \beta z)\hat{x}\,\frac{\mathrm{V}}{\mathrm{m}}$,

i. the corresponding phasors are given by

$$\begin{aligned}\tilde{\mathbf{E}}_1 &= \hat{x}4e^{-j\beta z}\,\frac{\mathrm{V}}{\mathrm{m}} \\ \tilde{\mathbf{H}}_1 &= \hat{y}\frac{4}{\eta_o}e^{-j\beta z}\,\frac{\mathrm{A}}{\mathrm{m}}.\end{aligned}$$

ii. It is **linearly polarized** in $\hat{x}$-direction.

b) For the wave described by $\mathbf{E}_2 = 5\cos(\omega t - \beta y)\hat{x} + 5\sin(\omega t - \beta y)\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}}$,

i. the corresponding phasors are given by

$$\tilde{\mathbf{E}}_2 = \hat{x}5e - \hat{z}5je^{-j\beta y} = 5e^{-j\beta y}\left(\hat{x} - j\hat{z}\right)\frac{\mathrm{V}}{\mathrm{m}}$$

$$\tilde{\mathbf{H}}_2 = \frac{5}{\eta_o}e^{-j\beta y}\left(-\hat{z} - j\hat{x}\right)\frac{\mathrm{A}}{\mathrm{m}}.$$

ii. Given that the wave propagates along $\hat{y}$ direction, it is seen that the wave is **left-hand-circularly** polarized.

c) For the wave described by $\mathbf{H}_3 = 2\cos(\omega t + \beta z + \frac{\pi}{3})\hat{x} + 2\sin(\omega t + \beta z - \frac{\pi}{6})\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}}$,

i. the corresponding phasors are given by

$$\tilde{\mathbf{H}}_3 = \hat{x}2e^{j\beta z}e^{j\frac{\pi}{3}} - \hat{y}2e^{j\beta z}e^{j\left(\frac{\pi}{2} - \frac{\pi}{6}\right)} = 2e^{j\left(\beta z + \frac{\pi}{3}\right)}\left(\hat{x} - \hat{y}\right)\frac{A}{\mathrm{m}}$$

$$\tilde{\mathbf{E}}_3 = 2\eta_o e^{j\left(\beta z + \frac{\pi}{3}\right)}\left(\hat{y} + \hat{x}\right)\frac{V}{\mathrm{m}}.$$

ii. Thus the wave is **linearly polarized** in $\frac{\hat{x}+\hat{y}}{\sqrt{2}}$ direction.

d) For the wave described by $\mathbf{H}_4 = 4\cos(\omega t - \beta x)\hat{z} - 3\sin(\omega t - \beta x)\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}}$,

i. the corresponding phasors are given by

$$\tilde{\mathbf{H}}_4 = \hat{z}4e^{-j\beta x} + \hat{y}3je^{-j\beta x} = e^{-j\beta x}\left(4\hat{z} + 3j\hat{y}\right)\frac{\mathrm{A}}{\mathrm{m}}$$

$$\tilde{\mathbf{E}}_4 = \eta_o e^{-j\beta x}\left(4\hat{y} - 3j\hat{z}\right)\frac{V}{\mathrm{m}}.$$

ii. Since the two components have different magnitudes, the wave is **elliptical** polarized.

e) For the wave described by $\mathbf{H}_5 = 2\sin(\omega t + \beta y)\hat{x} - 2\sin(\omega t + \beta y - \frac{\pi}{4})\hat{z}\,\frac{\mathrm{A}}{\mathrm{m}}$,

i. the corresponding phasors are given by

$$\tilde{\mathbf{H}}_5 = -\hat{x}2je^{j\beta y} + \hat{z}2je^{j\beta y - j\frac{\pi}{4}} = 2je^{j\beta y}\left(-\hat{x} + e^{-j\frac{\pi}{4}}\hat{z}\right)\frac{\mathrm{A}}{\mathrm{m}}$$

$$\tilde{\mathbf{E}}_5 = 2\eta_o je^{j\beta y}\left(\hat{z} + e^{-j\frac{\pi}{4}}\hat{x}\right)\frac{V}{\mathrm{m}}$$

ii. Since the phase angle between $\hat{x}$ and $\hat{z}$ components of $\tilde{\mathbf{E}}_5$ is $\frac{\pi}{4}$, not an integer multiple of $\frac{\pi}{2}$, it is seen that the wave is **elliptical** polarized.

4.

a) The phasors form of surface current densities are

$$\tilde{\mathbf{J}}_{s1} = \hat{z} J_1 e^{-j\phi} \, \frac{\mathrm{A}}{\mathrm{m}} \quad (x = 0),$$

$$\tilde{\mathbf{J}}_{s2} = \hat{y} J_2 \, \frac{\mathrm{A}}{\mathrm{m}} \quad (x = \frac{\lambda}{4}).$$

Then, recalling $\beta = \frac{2\pi}{\lambda}$, the corresponding electric fields propagating in the region $x > \frac{\lambda}{4}$ are given by

$$\begin{aligned} \tilde{\mathbf{E}}_1 &= \hat{z} \frac{\eta_o}{2} J_1 e^{j(-\beta x - \phi + \pi)} \, \frac{\mathrm{V}}{\mathrm{m}}, \\ \tilde{\mathbf{E}}_2 &= \hat{y} \frac{\eta_o}{2} J_2 e^{j(-\beta x + \frac{3\pi}{2})} \, \frac{\mathrm{V}}{\mathrm{m}}. \end{aligned}$$

The total field is $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_1 + \tilde{\mathbf{E}}_2 = \frac{\eta_o}{2} J e^{j(-\beta x + \pi)} (\hat{y} e^{j\frac{\pi}{2}} + \hat{z} e^{-j\phi})$ where $J = J_1 = J_2$.

i. The direction of propagation is $+\hat{x}$ and the wave is **RHC**. Then, the $\hat{y}$ component needs to lead the $\hat{z}$ component by 90^o. Therefore

$$\phi + \frac{\pi}{2} = \frac{\pi}{2} + 2n\pi \rightarrow \phi = 2n\pi$$

where n is an arbitrary integer. The electric field phasor for the region will be $\tilde{\mathbf{E}} = \frac{\eta_o}{2} J e^{j(-\beta x + \pi)} (\hat{y} e^{j\frac{\pi}{2}} + \hat{z} e^{j2n\pi}) \, \frac{\mathrm{V}}{\mathrm{m}}$.

ii. To have **LHC** polarization, the $\hat{z}$ component needs to lead by 90^o the $\hat{y}$ component. Therefore

$$\phi + \frac{\pi}{2} = -\frac{\pi}{2} + 2n\pi \rightarrow \phi = 2n\pi - \pi,$$

where n is an arbitrary integer. The electric field phasor for the region will be $\tilde{\mathbf{E}} = \frac{\eta_o}{2} J e^{j(-\beta x + \pi)} (\hat{y} e^{j\frac{\pi}{2}} - \hat{z} e^{j2n\pi}) \, \frac{\mathrm{V}}{\mathrm{m}}$.

iii. To have linear polarization, the $\hat{z}$ component needs to be in phase with the $\hat{y}$ component or off by 180^o. Thus, we write

$$\phi + \frac{\pi}{2} = n\pi \rightarrow \phi = n\pi - \frac{\pi}{2}$$

where n is an arbitrary integer. The electric field phasor for the region will be $\tilde{\mathbf{E}} = \frac{\eta_o}{2} J e^{j(-\beta x + \pi)} (\hat{y} e^{j\frac{\pi}{2}} - \hat{z} j e^{jn\pi}) \, \frac{\mathrm{V}}{\mathrm{m}}$.

b) The corresponding magnetic field is

$$\tilde{\mathbf{H}} = \frac{1}{2} J e^{j(-\beta x + \pi)} (\hat{z} e^{j\frac{\pi}{2}} - \hat{y} e^{-j\phi}) \, \frac{\mathrm{A}}{\mathrm{m}},$$

where $J = J_1 = J_2 = 1\,\mathrm{A/m}$. Therefore, the time-averaged Poynting vector is

$$\begin{aligned} \langle \mathbf{S} \rangle &= \frac{1}{2} \mathbf{Re} \left\{ \tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* \right\} \\ &= \frac{1}{2} \frac{\eta_o}{4} J^2 Re\{2\hat{x}\} \\ &= \frac{\eta_o}{4} \hat{x} \end{aligned}$$

This result does not depend on the angle ϕ, therefore the time-averaged Poynting vector

$$\langle \mathbf{S} \rangle = \hat{x}\, 30\pi \, \frac{\mathrm{W}}{\mathrm{m}^2}$$

c) If J_2 =0, then $\tilde{\mathbf{E}}_2 = 0$, and $\tilde{\mathbf{H}}_2 = 0$. We are left with:

$$\tilde{\mathbf{E}}_1 = \hat{z}\frac{\eta_o}{2} J_1 e^{j(-\beta x - \phi + \pi)} \, \frac{\mathrm{V}}{\mathrm{m}}$$

and

$$\tilde{\mathbf{H}}_1 = -\hat{y}\frac{1}{2} J_1 e^{j(-\beta x - \phi + \pi)} \, \frac{\mathrm{V}}{\mathrm{m}}$$

therefore the time-averaged Poynting vector is

$$\begin{aligned} \langle \mathbf{S} \rangle &= \frac{1}{2}\mathbf{Re}\left\{ \tilde{\mathbf{E}}_1 \times \tilde{\mathbf{H}}_1^* \right\} = \frac{1}{2}\frac{\eta_o}{4} J_1^2 \hat{x} \\ &= \hat{x}\frac{\eta_o}{8} J^2 = \hat{x}\, 15\pi \, \frac{\mathrm{W}}{\mathrm{m}^2}. \end{aligned}$$

d) Note that the two currents in (b) generally produces circular polarized waves, while the single current sheet in (c) produces linear polarized waves. From the results of (b) and (c), we can see that in case of circularly polarized waves the power content is twice that of a linearly polarized wave field of an equal instantaneous peak electric field magnitudes.

5. When a wave is incident on a boundary between two different media, a reflected wave is produced. In addition, if the second medium is not a perfect conductor, a transmitted wave is set up. Together, these waves satisfy the boundary conditions at the interface of the two media. We shall assume that a wave is incident from medium 1 (z < 0) onto the interface, thereby setting up a reflected wave in that medium, and a transmitted wave in medium 2 (z > 0).

a) The reflection coefficient for this boundary is defined from lecture:

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

and the transmission coefficient:

$$\tau = 1 + \Gamma = 1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

b) Given an incident field of $E(z,t) = \hat{y}E_0 cos(\omega t - \beta_1 z)$, this can be written in phasor form as:

$$\tilde{\mathbf{E}}_i = \hat{y}E_0 e^{-j\beta_1 z}$$

The reflected field is defined as

$$\tilde{\mathbf{E}}_r = \hat{y}\Gamma \left|\tilde{E}_i\right| e^{j\beta_1 z} = \hat{y}\Gamma E_0 e^{j\beta_1 z}$$

Similarly, the transmitted electric field is

$$\tilde{\mathbf{E}}_t = \hat{y}\tau \left|\tilde{E}_i\right| e^{-j\beta_2 z} = \hat{y}\tau E_0 e^{-j\beta_2 z}$$

The magnetic field's incident, reflected and transmitted components will have their amplitudes scaled by the characteristic impedance of each region and will be oriented to satisfy $\hat{z} \times \tilde{\mathbf{E}}_i = \eta_1 \tilde{\mathbf{H}}_i$, $\hat{z} \times \tilde{\mathbf{E}}_t = \eta_2 \tilde{\mathbf{H}}_t$, and $(-\hat{z}) \times \tilde{\mathbf{E}}_r = \eta_1 \tilde{\mathbf{H}}_r$. Therefore, we can write

$$\tilde{\mathbf{H}}_i = -\hat{x}\frac{E_0}{\eta_1} e^{-j\beta_1 z}$$

$$\tilde{\mathbf{H}}_r = \hat{x}\frac{\Gamma E_0}{\eta_1} e^{j\beta_1 z}$$

and

$$\tilde{\mathbf{H}}_t = -\hat{x}\frac{\tau E_0}{\eta_2} e^{-j\beta_2 z}$$

c) The complex Poynting vectors are given by

$$\tilde{\mathbf{S}}_i = \tilde{\mathbf{E}}_i \times \tilde{\mathbf{H}}_i^* = (\hat{y} \times (-\hat{x}))\frac{E_0^2}{\eta_1} = \hat{z}\frac{E_0^2}{\eta_1}$$

$$\tilde{\mathbf{S}}_r = \tilde{\mathbf{E}}_r \times \tilde{\mathbf{H}}_r^* = (\hat{y} \times \hat{x})\frac{\Gamma^2 E_0^2}{\eta_1} = -\hat{z}\frac{\Gamma^2 E_0^2}{\eta_1}$$

and

$$\tilde{\mathbf{S}}_t = \tilde{\mathbf{E}}_t \times \tilde{\mathbf{H}}_t^* = (\hat{y} \times (-\hat{x}))\frac{\tau^2 E_0^2}{\eta_2} = \hat{z}\frac{\tau^2 E_0^2}{\eta_2}$$

The time average Poynting vectors are therefore given by

$$\langle \mathbf{S}_i \rangle = \frac{1}{2}\text{Re}\left[\tilde{\mathbf{S}}_i\right] = \hat{z}\frac{E_0^2}{2\eta_1}$$

$$\langle \mathbf{S}_r \rangle = \frac{1}{2}\text{Re}\left[\tilde{\mathbf{S}}_r\right] = -\hat{z}\frac{\Gamma^2 E_0^2}{2\eta_1}$$

and

$$\langle \mathbf{S}_t \rangle = \frac{1}{2}\text{Re}\left[\tilde{\mathbf{S}}_t\right] = \hat{z}\frac{\tau^2 E_0^2}{2\eta_2}$$

d) The conservation of energy states that the power incident on the boundary must be equal to the power leaving the boundary. Because we are considering uniform plane waves, the power density (given by the Poynting vector) is constant over the x-y plane, so equivalently we can say that the incident power density must equal the outgoing power density. This can be expressed as

$$|\langle \mathbf{S}_i \rangle| = |\langle \mathbf{S}_r \rangle| + |\langle \mathbf{S}_t \rangle|$$

To verify this for the given case, we can first calculate Γ and τ, noting that $\eta_1 = \eta_0$ (free space) and $\eta_2 = \sqrt{\mu_0/(2\epsilon_0)} = \eta_0/\sqrt{2}$, as

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_0/\sqrt{2} - \eta_0}{\eta_0/\sqrt{2} + \eta_0}$$

$$= \frac{1-\sqrt{2}}{1+\sqrt{2}} \approx -0.1715$$

Similarly

$$\tau = 1 + \Gamma \approx 0.8285$$

Next, we can insert these values to check that the conservation equation holds:

$$|\langle \mathbf{S}_i \rangle| = \frac{E_0^2}{2\eta_1} \approx 0.00132 E_0^2$$

$$|\langle \mathbf{S}_r \rangle| = \frac{\Gamma^2 E_0^2}{2\eta_1} \approx 3.882 \times 10^{-5} E_0^2$$

and

$$|\langle \mathbf{S}_t \rangle| = \frac{\tau^2 E_0^2}{2\eta_2} \approx 0.001287 E_0^2$$

These values can be inserted into the conservation equation to check that

$$|\langle \mathbf{S}_i \rangle| = \frac{E_0^2}{2\eta_1} = 0.00132 E_0^2$$

and

$$|\langle \mathbf{S}_r \rangle| + |\langle \mathbf{S}_t \rangle| = 3.882 \times 10^{-5} E_0^2 + 0.001287 E_0^2 = 0.00132 E_0^2 = |\langle \mathbf{S}_i \rangle|$$

confirming that conservation of energy is obeyed.

e) We will prove that conservation of energy is obeyed for arbitrary values of η_1 and η_2, corresponding to any lossless medium. First insert the definitions of the time-average power density of part (c) into the conservation equation of part (d):

$$|\langle \mathbf{S}_i \rangle| = \frac{E_0^2}{2\eta_1} = \frac{\Gamma^2 E_0^2}{2\eta_1} + \frac{\tau^2 E_0^2}{2\eta_2} = |\langle \mathbf{S}_r \rangle| + |\langle \mathbf{S}_t \rangle|$$

Next, we can divide both sides of the middle equation by the common factor of $E_0^2/2$ to reveal

$$\frac{1}{\eta_1} = \frac{\Gamma^2}{\eta_1} + \frac{\tau^2}{\eta_2}$$

an equivalent condition that must hold to satisfy conservation of energy. To show that this holds regardless of the impedances, we can substitute in the definitions of the reflection and transmission coefficients from (a) as

$$\frac{1}{\eta_1} = \frac{1}{\eta_1}\left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)^2 + \frac{1}{\eta_2}\left(\frac{2\eta_2}{\eta_2 + \eta_1}\right)^2$$

Multiply both sides by $\eta_1\eta_2$ to gain common denominators on the right hand side:

$$\eta_2 = \frac{\eta_2\left(\eta_2 - \eta_1\right)^2 + 4\eta_1\eta_2}{\left(\eta_2 + \eta_1\right)^2} = \eta_2 \frac{\eta_2^2 + 2\eta_2\eta_1 + \eta_1^2}{\eta_2^2 + 2\eta_2\eta_1 + \eta_1^2} = \eta_2$$

implying that the conservation of energy equation holds regardless of the values of η_1, η_2.