University of Illinois Summer 2017

ECE 313: Final Exam

Friday, August 4, 2017, 8-10 a.m.
ECEB 3017

Name: (in **BLOCK CAPITALS**) Solutions

NetID: ______________________

Signature: ______________________

Instructions

This exam is closed book and closed notes except that two 8.5"×11" sheets of notes is permitted: both sides may be used. No electronic equipment (cell phones, etc.) allowed.

The exam consists of 7 problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading	
1. 14 points	
2. 10 points	
3. 14 points	
4. 14 points	
5. 25 points	
6. 13 points	
7. 10 points	
Total (100 points)	

1. **[14 points]** A message source that produces a sequence of 4 bits of information is equally likely to be in one of three states: S_1, S_2, and S_3. In state S_k, the 4 bit sequence can have at most k 1's, with all such 4 bit sequences being equally likely. For example, in state S_1, the source can produce the sequences 0000, 1000, 0100, 0010, 0001 with each of sequences being produced with probability $\frac{1}{5}$.

   Let A be event that the sequence 1010 was produced by the source.

   (a) **[4 points]** Find $P(A|S_2)$.

   $S_2 \rightarrow$ at most 2 1's out of 4

   $$\Rightarrow |S_2| = \underbrace{\binom{4}{0}}_{0 \text{ 1's}} + \underbrace{\binom{4}{1}}_{1 \text{ 1's}} + \underbrace{\binom{4}{2}}_{2 \text{ 1's}} = 1+4+6 = 11$$

   $$\Rightarrow P(A|S_2) = \frac{1}{|S_2|} = \boxed{\frac{1}{11}}$$

   using reduced sample space to S_2.

   (b) **[6 points]** Find $P(A)$.

   Using total probability

   $|S_3| = |S_2| + \binom{4}{3} = 11+4 = 15$

   $$P(A) = \underbrace{P(A|S_1)}_{=0} P(S_1) + \underbrace{P(A|S_2)}_{=\frac{1}{11}} P(S_2) + \underbrace{P(A|S_3)}_{=\frac{1}{15}} P(S_3)$$

   $$= \frac{1}{11}\left(\frac{1}{3}\right) + \left(\frac{1}{15}\right)\left(\frac{1}{3}\right) = \boxed{\frac{26}{495}}$$

   (c) **[4 points]** Find the probability that the source was in state S_2 given event A.

   $$P(S_2|A) = \frac{P(A|S_2)\,P(S_2)}{P(A)} = \frac{\frac{1}{11}\left(\frac{1}{3}\right)}{\frac{26}{495}} = \boxed{\frac{15}{26}}$$

   by Bayes rule

2. **[10 points]** Consider a binary hypotheses testing problem with observation X. Under H_0, $X \sim \textit{Binomial}\,(72, \frac{1}{3})$, while under H_1, $X \sim \textit{Binomial}\,(72, \frac{2}{3})$. It is known that $\pi_0 = 0.9$. $\Rightarrow \pi_1 = 0.1$

   (a) **[6 points]** Determine the MAP decision rule. Express the rule in terms of k as simply as possible. In case of a tie in likelihoods, declare H_1 to be the hypothesis.

$$\Lambda(k) = \frac{p_1(k)}{p_0(k)} = \frac{\binom{72}{k}\left(\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{72-k}}{\binom{72}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{72-k}} \geq \frac{\pi_0}{\pi_1} = \frac{0.9}{0.1} = 9$$

declare H_1

$2^0=1$

$2^1=2$

$2^2=4$

$2^3=8$

$2^4=16$

$$2^{2k-72} \geq 9 \Rightarrow 2k-72 \geq 4$$

$$k \geq 38$$

$\Rightarrow$ $\boxed{\text{declare } H_1 \text{ if } k \geq 38 \quad H_0 \text{ else}}$

   (b) **[4 points]** Express the approximate value of $p_{false\ alarm}$ for the MAP rule in terms of the Q function, where $Q(c) = \int_c^\infty \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)du$. (To be definite, don't use the continuity correction.)

$$P_{fa} = P\{\text{declare } H_1 | H_0\} = P\{X \geq 38\}$$

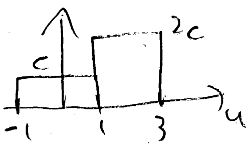
$X \sim$ Binomial $(72, \frac{1}{3})$ under $H_0 \Rightarrow \mu_X = \frac{72}{3} = 24$

$$\sigma_X^2 = 72\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right) = 16$$

$$\Rightarrow P_{fa} \approx Q\left(\frac{38-24}{\sqrt{16}}\right) = \boxed{Q\left(\frac{7}{2}\right)}$$

3. **[14 points]** Let c be a constant and X be a random variable with pdf

$$f_X(u) = \begin{cases} c & u \in [-1, 1) \\ 2c & u \in [1, 3] \\ 0 & \text{else.} \end{cases}$$



You may leave your answers to this problem in terms of c, except for part (a).

(a) **[3 points]** Obtain the value of the constant c.

$\int_{-\infty}^{\infty} f_X(u)\,du = 1$ = area under curve $= 2c + 2(2c)$

$1 = 6c \Rightarrow \boxed{c = \frac{1}{6}}$

(b) **[6 points]** Determine the CDF $F_X(u)$ for all u.

$F_X(u) = \int_{-\infty}^{u} f_X(s)\,ds$

$$= \begin{cases} 0 & u < -1 \\ \int_{-1}^{u} c\,ds = c(u+1) & -1 < u < 1 \\ \int_{-1}^{1} c\,ds + \int_{1}^{u} 2c\,ds = 2c + 2c(u-1) = 2cu & 1 < u < 3 \\ 1 & u > 3 \end{cases}$$

$$\boxed{F_X(u) = \begin{cases} 0 & u < -1 \\ c(u+1) & -1 < u < 1 \\ 2cu & 1 < u < 3 \\ 1 & u > 3 \end{cases}}$$

(c) **[5 points]** Obtain $E[sin(X)]$ *You may leave your answer in terms of sin and cos.*

$E[\sin(X)] = \int_{-\infty}^{\infty} \sin(u) f_X(u)\,du = \int_{-1}^{1} \sin(u) c\,du + \int_{1}^{3} \sin(u) 2c\,du$

LOTUS

$= c\left[-\cos u\Big|_{-1}^{1} - 2\cos u\Big|_{1}^{3}\right] =$

$= -c\left[\underbrace{\cos(1) - \cos(-1)}_{=0} + 2(\cos(3) - \cos(1))\right] =$

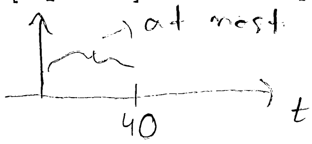
$\boxed{= 2c[\cos(1) - \cos(3)]}$

3

4. **[14 points]** Joe fishes with an unusual strategy: he waits for three fish to jump, then throws a net immediately after seeing the third fish jump. Fish jump according to a Poisson process with parameter $\lambda = 0.1$ jumps per minute. Assume that Joe begins waiting at time zero; the first fish jumps at time T_1, the second fish jumps at time T_2 and the third fish jumps at time T_3. *You may leave powers of e in your answer.*

(a) **[5 points]** What's the probability that $T_3 \geq 40$ minutes?

→ at most 2 fishes

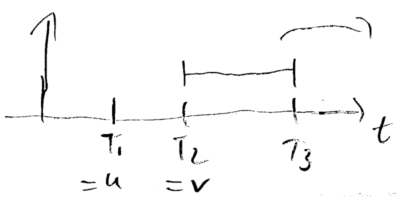


$N_{40} \sim \text{Poisson}(0.1(40)) = \text{Poisson}(4)$

$$P\{T_3 \geq 40\} = P\{N_{40} \leq 2\} = P\{N_{40}=0\} + P\{N_{40}=1\} + P\{N_{40}=2\}$$

$$= e^{-4}\frac{4^0}{0!} + e^{-4}\frac{4^1}{1!} + e^{-4}\frac{4^2}{2!} = \boxed{13e^{-4}}$$

(b) **[5 points]** Suppose that Joe sees the first fish jump at time $T_1 = u > 0$ minutes and the second fish jump at time $T_2 = v > u > 0$ minutes. What is the conditional pdf $f_{T_3|T_1,T_2}(t|u,v)$ for all t?



→ Exponential(λ)

⇒ conditionally $T_3 = v + \text{Exponential}(0.1)$

$$\Rightarrow \boxed{f_{T_3|T_1,T_2}(t|u,v) = \begin{cases} 0.1e^{-0.1(t-v)} & t \geq v \\ 0 & \text{else} \end{cases}}$$

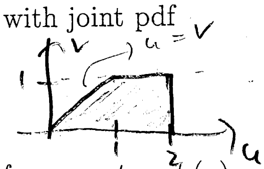
(c) **[4 points]** What is $E[T_3|T_1 = u, T_2 = v]$, for $0 < u < v$?

From (b)

$$E[T_3|T_1=u, T_2=v] = \frac{1}{\lambda} + v = \boxed{10+v}$$

5. **[25 points]** Let X and Y be jointly continuous random variables with joint pdf

$$f_{X,Y}(u,v) = \begin{cases} cv & v \in [0,1], u \in [v,2] \\ 0 & \text{else,} \end{cases}$$



for a constant c. *You may leave this problem's answers in terms of c, except part (a).*

(a) **[3 points]** Determine the value of the constant c.

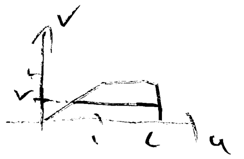
$$\iint_{\mathbb{R}^2} f_{X,Y}(u,v)\,du\,dv = 1 = \int_0^1 \int_v^2 cv\,du\,dv = c\int_0^1 v \int_v^2 du\,dv$$

$$= c\frac{2}{3} \implies \boxed{c = \frac{3}{2}}$$

(b) **[4 points]** Obtain the marginal pdf $f_Y(v)$ for all v.

$$f_Y(v) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,du = \int_v^2 cv\,du = cv(2-v)$$

$$\boxed{= \begin{cases} cv(2-v) & v \in [0,1] \\ 0 & \text{else} \end{cases}}$$



(c) **[4 points]** Obtain the conditional pdf $f_{X|Y}(u|v)$ for all u and v.

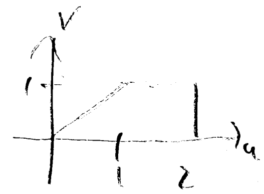
$f_{X|Y}(u|v) = \dfrac{f_{X,Y}(u,v)}{f_Y(v)}$ if $f_Y(v)$, otherwise undefined

$$\implies \boxed{f_{X|Y}(u|v) = \begin{cases} \text{undefined} & v \notin [0,1] \\ \frac{1}{2-v} & v \in [0,1], u \in [v,2] \\ 0 & \text{else} \end{cases}}$$

$$\frac{cv}{cv(2-v)} = \frac{1}{2-v}$$

$\implies$ Uniform$(v,2)$

(d) **[8 points]** Recall that



$$f_{X,Y}(u,v) = \begin{cases} cv & v \in [0,1], u \in [v,2] \\ 0 & \text{else,} \end{cases}$$

Obtain the best unconstrained MMSE estimator of X from Y, $\hat{X} = g^*(Y)$. *Notice that you are not estimating Y from X.*

Notice, either from (c) or from $f_{X,Y}$ that conditional on $Y=v$, $X \sim \text{Uniform}(v,2)$

$\Rightarrow g^*(y) = E[X|y] = \frac{y+2}{2} = \boxed{\frac{1}{2}y+1}$

(e) **[6 points]** Obtain the best linear MMSE estimator of X from Y, $\hat{X} = L^*(Y) = \hat{E}[X|Y]$. *Notice that you are not estimating Y from X.*

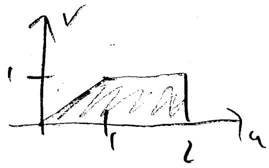
Notice from (d) that the best unconstrained estimator is linear, hence the best linear estimator must be the same line $\Rightarrow$ $L^*(y) = g^*(y) = \boxed{\frac{1}{2}y+1}$

You could also do the whole calculation for $L^*(y) = \frac{\text{Cov}(X,Y)}{\text{Var}(Y)}(y - \mu_Y) + \mu_X$

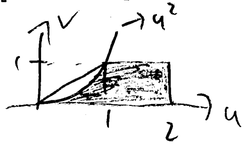
$\text{Cov}(X,Y) = \frac{19}{640}$, $\text{Var}(Y) = \frac{19}{320}$, $\mu_Y = \frac{5}{8}$, $\mu_X = \frac{21}{16}$

6. **[13 points]** Let X and Y be jointly uniform random variables with joint pdf

$$f_{X,Y}(u,v) = \begin{cases} \frac{2}{3} & v \in [0,1], u \in [v,2] \\ 0 & \text{else,} \end{cases}$$



(a) **[5 points]** Obtain $P\{Y - X^2 \leq 0\}$. $= P\{Y \leq X^2\}$



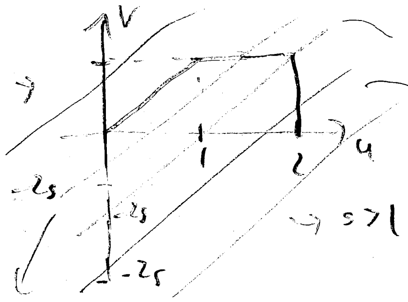
$$= \int_0^1 \int_{\sqrt{v}}^2 \frac{2}{3}\, du\, dv = \frac{2}{3} \int_0^1 \int_{\sqrt{v}}^2 du\, dv$$

$u^2 = v \Rightarrow u = \pm\sqrt{v}$

$$\boxed{= \frac{8}{9}}$$

(b) **[8 points]** Let $Z = \frac{1}{2}X - \frac{1}{2}Y$. Obtain the CDF $F_Z(s)$ for all s.

$F_Z(s) = P\{Z \leq s\} = P\{\frac{1}{2}X - \frac{1}{2}Y \leq s\} = P\{X - 2s \leq Y\}$



$s < 0$

$\to$ integrals will change after

$s \in (\frac{1}{2}, 1]$ $\to$ line @ $X = 2 \Rightarrow Y = 0$

$2 - 2s = 0 \to s = 1$

because line @ $X = 2 \Rightarrow Y = 1$

$2 - 2s = 1 \Rightarrow s = \frac{1}{2}$

$s > 1$

$s \in (0, \frac{1}{2}]$ $\to$ line @ $X = 0 \Rightarrow Y = 0 \Rightarrow s = 0$

$X - 2s = Y \Rightarrow X = Y + 2s$

For $s \in (0, \frac{1}{2})$ $F_Z(s) = \int_0^1 \int_v^{v+2s} \frac{2}{3}\, du\, dv = \frac{4s}{3}$

$$F_Z(s) = 1 - \int_0^{2-2s} \int_{v+2s}^{2} \frac{2}{3}\, du\, dv = 1 - \frac{4}{3}(1-s)^2$$

$$\Rightarrow \boxed{F_Z(s) = \begin{cases} 0 & s < 0 \\ \frac{4}{3}s & 0 < s < \frac{1}{2} \\ 1 - \frac{4}{3}(1-s)^2 & \frac{1}{2} < s < 1 \\ 1 & s > 1 \end{cases}}$$

7. **[10 points]** (2 points per answer)
No partial credit. In order to discourage guessing, 2 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Let X and Y be jointly random variables

TRUE FALSE

☐ ☒ If $E[X+Y] = E[X] + E[Y]$ then X and Y are independent.

This always holds, even if not independent.

☐ ☒ If $Var(X+Y) = Var(X) + Var(Y)$ then X and Y are independent.

This holds also if X,Y are just uncorrelated

$$Var(X+Y) = Var(X) + Var(Y) + \overset{0}{\cancel{2Cov(X,Y)}}$$

☐ ☒ If $f_{X,Y}(1, 2.5) = f_X(1)f_Y(2.5)$ then X and Y are independent.

This must hold everywhere on the plane, not just at one point.

$$f_{X,Y}(u,v) = f_X(u) f_Y(v)$$

☒ ☐ If X and Y are independent then $E[X^2 sin(Y)] = E[X^2]E[sin(Y)]$

If independent $f_{X,Y}(u,v) = f_X(u) f_Y(v)$

$$E[X^2 \sin Y] = \iint u^2 \sin v \, f_{X,Y}(u,v)\, du\, dv$$
$$= \int u^2 f_X(u)\, du \int \sin v \, f_Y(v)\, dv$$
$$= E[X^2] E[\sin Y]$$

☐ ☒ If $\hat{E}[Y|X] = aX + b$ then $\hat{E}[X|Y] = \frac{1}{a}Y - \frac{1}{a}b$

For example, if X, Y are independent and $\mu_X \neq \mu_Y$ then $\hat{E}[Y|X] = \mu_Y$ $\neq$ $\hat{E}[X|Y] = \mu_Y$