University of Illinois Summer 2017

ECE 313: Exam II

Thursday, July 20, 2017
5.15 - 6.30 p.m.
3015 ECEB

Name: (in **BLOCK CAPITALS**) Solutions

NetID: ______________________

Signature: ______________________

Instructions

This exam is closed book and closed notes except that one 8.5"×11" sheet of notes is permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

The exam consists of **five** problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

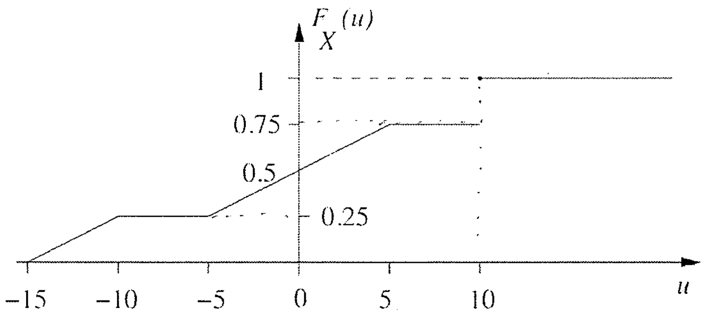
SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the blank page at the end of the exam.

Grading

1. 18 points ________
2. 20 points ________
3. 20 points ________
4. 22 points ________
5. 20 points ________

Total (100 points) ________

1. **[18 points]** Let X be a random variable with CDF plotted below.



(a) Obtain $P\{X < 10\}$. $= F_X(10^-) = \boxed{0.75}$

(b) Obtain $P\{X > -5\}$. $= 1 - F_X(-5) = 1 - 0.25 = \boxed{0.75}$

(c) Obtain $P\{X = 1\}$. $= F_X(1) - F_X(1^-) = \boxed{0}$

(no jump)

(d) Obtain $P\{|X| \leq 10\}$. $= P\{-10 \leq X \leq 10\} = F_X(10) - F_X(-10^-)$

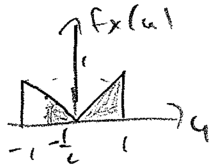
$= 1 - 0.25 = \boxed{0.75}$

(e) Determine if X is a discrete, continuos or mixed-type random variable, and explain why.

It is mixed-type because it has jumps and also strictly increasing and continuous portions.

2. **[20 points]** Let X have pdf $f_X(u) = |u|$ for $u \in [-1, 1]$ and zero else.

(a) Determine $P\{X > -\frac{1}{2}\}$. $= \int_{-\frac{1}{2}}^{\infty} f_X(u)\,du = \int_{-\frac{1}{2}}^{0} (-u)\,du + \int_{0}^{1} u\,du = \frac{5}{8}$



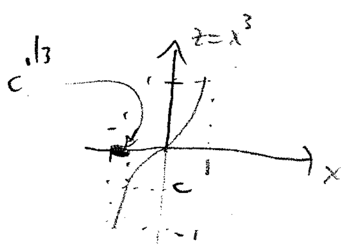
$= (\text{area under } f_X(u) \text{ right of } -\tfrac{1}{2})$

$= \frac{\frac{1}{2}\left(\frac{1}{2}\right)}{2} + \frac{(1)(1)}{2} = \boxed{\frac{5}{8}}$

(b) Determine $P\{X < 0 | X > -\frac{1}{2}\}$. $= \frac{P\{X<0, X>-\frac{1}{2}\}}{P\{X>-\frac{1}{2}\}} = \frac{P\{-\frac{1}{2}<X<0\}}{5/8}$

$= \frac{8}{5}\int_{-\frac{1}{2}}^{0} (-u)\,du = \boxed{\frac{1}{5}}$

(c) Let $Z = X^3$, determine its pdf $f_Z(c)$ for all c, and determine its mean, $E[Z]$.



$z \in [-1, 1]$

$F_Z(c) = P\{Z \le c\} = P\{X^3 \le c\} = P\{X \le c^{1/3}\}$

$= F_X(c^{1/3})$

$f_Z(c) = \frac{d}{dc} F_Z(c) = f_X(c^{1/3})\,\frac{1}{3}c^{-2/3}$

$$\boxed{f_Z(c) = \begin{cases} -c^{1/3}\,\frac{1}{3}c^{-2/3} = -\frac{1}{3}c^{-1/3} & c \in [-1, 0) \\ c^{1/3}\,\frac{1}{3}c^{-2/3} = \frac{1}{3}c^{-1/3} & c \in [0, 1] \\ 0 & \text{else} \end{cases}}$$

pdf of X is an even function with finite support using LOTUS : $\boxed{E[X^3] = 0}$

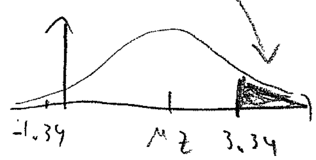
3. **[20 points]** The three parts of this problem are unrelated

(a) Let $X \sim N(-2, 9)$. Determine $P\{|X| < 3\}$ in terms of the Φ or Q functions.

$$P\{|X|<3\} = P\{-3<X<3\} = P\left\{\frac{-3-(-2)}{\sqrt{9}} < \frac{X-(-2)}{\sqrt{9}} < \frac{3-(-2)}{\sqrt{9}}\right\}$$

$$= P\{-\tfrac{1}{3} < \underbrace{Y}_{\sim N(0,1)} < \tfrac{5}{3}\} = \boxed{\Phi\left(\tfrac{5}{3}\right) - \Phi\left(-\tfrac{1}{3}\right) = Q\left(-\tfrac{1}{3}\right) - Q\left(\tfrac{5}{3}\right)}$$

(b) Let $Z \sim N(\mu_Z, \sigma_Z^2)$. It is known that $P\{Z < -1.34 \text{ or } Z > 3.34\} = 0.242$ and that $P\{Z > 3.34\} = 0.121$. Determine μ_Z.



mutually exclusive

$$P\{Z<-1.34 \text{ or } Z>3.34\} = P\{Z<-1.34\} + P\{Z>3.34\}$$

$$0.242 = P\{Z<-1.34\} + 0.121$$

$$\Rightarrow P\{Z<-1.34\} = 0.242 - 0.121 = 0.121 = P\{Z>3.34\}$$

so μ_Z is halfway between -1.34 and 3.34

$$\Rightarrow \mu_Z = \frac{-1.34+3.34}{2} = \boxed{1 = \mu_Z}$$

(c) Let $Y \sim \mathit{Binomial}(100, 0.5)$. Use the Gaussian approximation with continuity correction to determine $P\{Y < 60\}$ in terms of the Φ or Q functions.

$\mu_Y = np = 100\left(\tfrac{1}{2}\right) = 50 \qquad \sigma_Y^2 = np(1-p) = 50\left(\tfrac{1}{2}\right) = 25$

Y is integer-valued, hence

$$P\{Y<60\} = P\{Y \le 59\} \approx P\{X \le 59+0.5\}$$

$$= P\left\{\frac{X-\mu_Y}{\sigma_Y} \le \frac{59.5-50}{\sqrt{25}}\right\} = \boxed{\Phi(1.9)}$$

4. **[22 points]** Suppose the number of visitors to a popular website follows a Poisson process with rate 4 visitors per second. NOTE: you do not need to simplify the answers to this problem.

(a) What is the probability of exactly three visitors each minute in four consecutive minutes?

$$P\{N_1 - N_0 = 3, N_2 - N_1 = 3, N_3 - N_2 = 3, N_4 - N_3 = 3\} = \left(P\{\underbrace{N_1 - N_0}_{\sim \text{Poisson}(4(60))} = 3\}\right)^4$$

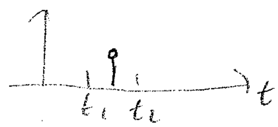
time in minutes

because of independence of non-overlapping intervals

$4(60) = 240$

$$\boxed{= \left[e^{-240}\frac{(240)^3}{3!}\right]^4}$$

(b) Let $0 < t_1 < t_2$, what is the probability that the first visitor arrives between t_1 and t_2? Assume that both t_1 and t_2 are measured in seconds.



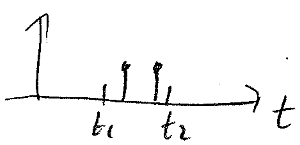
Arrival of 1st visitor is $Exp(\lambda)$

$$\Rightarrow P\{t_1 < T_1 < t_2\} = F_{T_1}(t_2) - F_{T_1}(t_1)$$

$$= F^c_{T_1}(t_1) - F^c_{T_1}(t_2)$$

$$\boxed{= e^{-4t_1} - e^{-4t_2}}$$

(c) Let $0 < t_1 < t_2$, what is the probability that the second visitor arrives between t_1 and t_2 given that no visitors arrived before t_1? Assume that both t_1 and t_2 are measured in seconds.



Need at least 2 visitors between t_1 and t_2 for this to occur.

$$P\{t_1 < T_2 < t_2 \mid N_{t_1} = 0\} = P\{N_{t_2} - N_{t_1} \geq 2\}$$

$$= 1 - P\{N_{t_2} - N_{t_1} < 2\} =$$

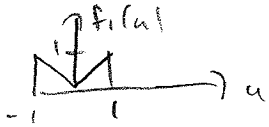
$$= 1 - \left[e^{-\lambda(t_2 - t_1)}\frac{(\lambda(t_2 - t_1))^0}{0!} + e^{-\lambda(t_2 - t_1)}\frac{(\lambda(t_2 - t_1))^1}{1!}\right]$$

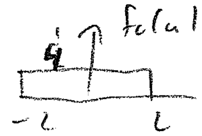
$$\boxed{= 1 - e^{-4(t_2 - t_1)}\left[1 + 4(t_2 - t_1)\right]}$$

5. **[20 points]** Suppose under hypothesis H_1, X has pdf $f_1(u) = |u|$ for $u \in [-1, 1]$ and zero else; while under hypothesis H_0, X is uniform on $[-2, 2]$. Let $\pi_0 = \frac{2}{3}$.

$\rightarrow \pi_1 = \frac{1}{3}$

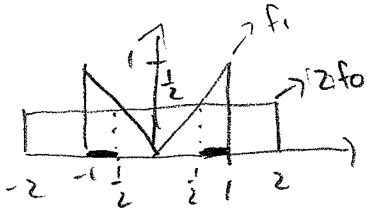
(a) Obtain the MAP decision rule.





$$\frac{\pi_0}{\pi_1} = 2$$

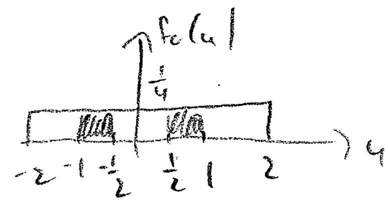
compare $f_1(u) > 2 f_0(u)$ declare H_1



$$\boxed{\begin{array}{l} \text{declare } H_1 \text{ if } X \in [-1, -\tfrac{1}{2}] \cup [\tfrac{1}{2}, 1] \\ H_0 \text{ else} \end{array}}$$

(b) Obtain $p_{false\ alarm}$ for the MAP rule.

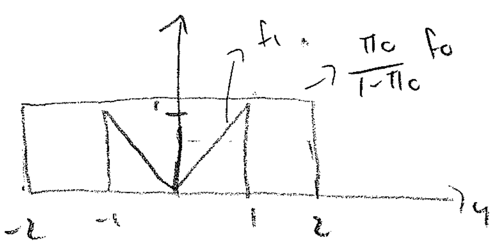
$$P_{fa} = P\{\text{declare } H_1 \mid H_0\} = P\{X \in [-1, -\tfrac{1}{2}] \cup [\tfrac{1}{2}, 1]\}$$



$$= 2\left(\tfrac{1}{2}\right)\tfrac{1}{4} = \boxed{\tfrac{1}{4}}$$

(c) Determine the value(s) of π_0 for which H_0 would always be chosen. to declare H_0

From (a) need to compare $f_1(u) < \frac{\pi_0}{1-\pi_0} f_0(u)$



Highest point in $f_1(u)$ is @ $u = \pm 1$.

$$\Rightarrow f_1(1) < \frac{\pi_0}{1-\pi_0} f_0(1)$$

$$1 < \frac{\pi_0}{1-\pi_0}\left(\frac{1}{4}\right)$$

$$\Rightarrow \pi_0 > \frac{4}{5} \Rightarrow \boxed{\frac{4}{5} < \pi_0 \leq 1}$$