University of Illinois — Summer 2017

ECE 313: Exam I

Thursday, July 6, 2017
5.15 - 6.30 p.m.
3015 ECEB

Name: (in BLOCK CAPITALS) Solutions

NetID: ______________________

Signature: ______________________

Instructions

This exam is printed **double-sided**, so make sure to look at all problems and both sides of every sheet.

This exam is closed book and closed notes except that one 8.5"×11" sheet of notes is permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

The exam consists of **five** problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the blank page at the end of the exam.

Grading	
1. 15 points	
2. 18 points	
3. 20 points	
4. 21 points	
5. 26 points	
Total (100 points)	

1. **[15 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score. For each of the following statements, determine if it is true or false for any three events A, B and C in a common probability space.

TRUE	FALSE	
☐	☒	$P(A\|B) + P(A\|B^c) = 1$.

Suppose $A \subset B$ with $P(A) < P(B)$.
So $P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} < 1$ and $P(A|B^c) = 0$

☐	☒	$P(A\|B) + P(C\|B) = 1$.

Suppose $B \subset A \subset C$ so $P(A|B) = P(C|B) = 1$

☒	☐	If A and B are mutually exclusive, then $P(A \cup C\|B) = P(C\|B)$.

$$P(A \cup C|B) = \underbrace{P(A|B)}_{\substack{=0 \\ \text{because m.e.}}} + P(C|B) - \underbrace{P(AC|B)}_{\substack{=0 \\ \text{because m.e.}}}$$

☒	☐	If $E_1, \ldots, E_n$ is a partition of Ω, then $\sum_{i=1}^n \underbrace{P(E_i\|B)}_{=P(E_iB)/P(B)} = 1$.

$$\sum \frac{P(E_iB)}{P(B)} = \frac{1}{P(B)} \sum P(E_iB) = \frac{1}{P(B)} P\left((E_1 \cup E_2 \cdots \cup E_n)B\right) = \frac{P(B)}{P(B)}$$

☐	☒	If $E_1, \ldots, E_n$ is a partition of Ω and $P(B) < 1$, then $\sum_{i=1}^n P(E_i\|B) < 1$.

See above.

2. **[18 points]** Dilbert and Wally are playing a game where Wally flips a coin and Dilbert wins if Heads show, while Wally wins if tails show. The coin is biased with $P\{heads\} = p$, but Dilbert does not know that. After five successive wins by Wally, Dilbert thinks that something is not right, so he decides he should now flip the coin (instead of Wally). Let X be the number of total coin flips (Wally's plus Dilbert's) until Dilbert wins.

(a) Obtain the pmf of X

Once Dilbert starts flipping, the number of flips until he wins is Geometric(p). Let Y be that number, then

$$P_Y(k) = P\{Y=k\} = (1-p)^{k-1}p. \quad \text{for } k = 1, 2, 3, \dots$$

However, there are 5 flips already, hence the total number of flips has pmf

$$\boxed{P_X(k) = (1-p)^{k-1-5}p = (1-p)^{k-6}p \quad \text{for } k = 6, 7, 8, \dots}$$

(b) Obtain $E[sin(\pi X)]$.

X is integer valued, so $\sin(\pi X) = 0$ for all X.

Hence $\boxed{E[\sin(\pi X)] = 0}$

Could also do $E[\sin(\pi X)] = \sum_{k=6}^{\infty} \underbrace{\sin(\pi k)}_{=0}(1-p)^{k-6}p = 0$

(c) Obtain $E[3X^2]$.

By linearity of expectation

$$E[3X^2] = 3E[X^2] = 3E[(5+Y)^2] = 3E[25 + 10Y + Y^2]$$

$$= 75 + 30\underbrace{E[Y]}_{=\frac{1}{p}} + 3\underbrace{E[Y^2]}_{=\sigma_Y^2 + \mu_Y^2 = \frac{1-p}{p^2} + \frac{1}{p^2}} = 75 + 30\left(\frac{1}{p}\right) + 3\left[\frac{1-p}{p^2} + \frac{1}{p^2}\right]$$

$$\boxed{E[3X^2] = \frac{75p^2 + 27p + 6}{p^2}}$$

3. **[20 points]** A job arriving in a certain cloud computing system must be routed to one of eight servers. Due to the loads, the servers all have different rates. Routing the job to the server with the highest rate would require sampling the rates of all eight servers. Instead, the rates of three randomly selected distinct servers are sampled (all choices being equally likely) and the job is routed to the sampled server with the highest service rate.

   (a) Let $A = \{$the job is assigned to the server with the highest service rate$\}$. Find $P(A)$.

   Highest rate server must be among the three chosen, but the other two chosen can be any of the other seven servers. Hence

   $$P(A) = \frac{\binom{1}{1}\binom{7}{2}}{\binom{8}{3}} = \boxed{\frac{3}{8}}$$

   (b) Let $B = \{$the job is assigned to one of the four slowest servers$\}$. Find $P(B)$.

   The three chosen servers must be among the four slowest, hence

   $$P(B) = \frac{\binom{4}{3}}{\binom{8}{3}} = \frac{1}{14}$$

   (c) Let $C = \{$job is assigned to one of the two servers with the highest rates$\}$. Find $P(C)$.

   $C^c = \{$none of the two highest rate servers is chosen$\}$

   $$P(C) = 1 - P(C^c) = 1 - \frac{\binom{6}{3}}{\binom{8}{3}} = \boxed{\frac{9}{14}}$$

4. **[21 points]** Consider a binary hypothesis testing problem where X is Geometric(1/2) under hypothesis H_1 and *Geometric*(2/3) under hypothesis H_0. It is known that $\boxed{\pi_1 = 4/9.}$

$\therefore \pi_0 = \frac{5}{9}$

(a) Derive the MAP rule to decide on the underlying hypothesis.

$\Lambda(k) \geq \frac{\pi_0}{\pi_1} \rightarrow$ declare H_1

$$\frac{\left(\frac{1}{2}\right)^{k-1}\left(\frac{1}{2}\right)}{\left(\frac{1}{3}\right)^{k-1}\left(\frac{2}{3}\right)} \geq \frac{5}{4}$$

$$\left(\frac{3}{2}\right)^k \geq \frac{10}{4}$$

increases with k:

$k=1$ $\quad \left(\frac{3}{2}\right) \geq \frac{10}{4}$ X

$k=2$ $\quad \left(\frac{3}{2}\right)^2 = \frac{9}{4} \geq \frac{10}{4}$ X

$k=3$ $\quad \left(\frac{3}{2}\right)^3 = \frac{27}{8} \geq \frac{20}{8}$ ✓

$\Rightarrow$ choose H_1 if $X \geq 3$ and H_0 else

Can also do

$$k \ln\left(\frac{3}{2}\right) \geq \ln\left(\frac{5}{2}\right)$$

$$k \geq \frac{\ln\left(\frac{5}{2}\right)}{\ln\left(\frac{3}{2}\right)} = \frac{0.92}{0.41} \approx 2.25$$

(b) Determines the probability of false alarm for the MAP rule.

$P_{\text{false alarm}} = P\{\text{declare } H_1 \mid H_0\} = P\{X \geq 3 \mid H_0\}$

$= 1 - P\{X < 3 \mid H_0\} = 1 - \left[\frac{2}{3} + \frac{2}{3}\left(\frac{1}{3}\right)\right] = \boxed{\frac{1}{9}}$

(c) Determine the value(s) of π_1 for which H_1 would always be chosen.

$$\Lambda(k) \geq \frac{1-\pi_1}{\pi_1}$$

$\left(\frac{3}{2}\right)^k \geq 2\left(\frac{\pi_0}{\pi_1}\right)$ from (a) and increases with k,

hence need to check for smallest k, which is $k=1$.

$\left(\frac{3}{2}\right) \geq 2\left(\frac{1-\pi_1}{\pi_1}\right) \Rightarrow 3\pi_1 \geq 4 - 4\pi_1 \Rightarrow \pi_1 \geq \frac{4}{7}$

$\Rightarrow \boxed{\frac{4}{7} \leq \pi_1 \leq 1}$

5. **[26 points]** Alice and Dilbert play the following game of dice. Each round of the game consists of Alice rolling one die and Dilbert rolling two dice. All the dice are fair. Here are the rules of this game:

- If Alice's number falls strictly between the two numbers rolled by Dilbert, then Alice wins and the game ends, e.g. if Alice rolls a 3 and Dilbert rolls a 1 and a 4.
- If Alice's number is equal to one of the numbers rolled by Dilbert, then it is a tie and the round is repeated until there is a winner, e.g if Alice rolls a 3 and Dilbert rolls a 1 and a 3.
- In all other cases, Dilbert wins and the game ends, e.g if Alice rolls a 3 and Dilbert rolls a 1 and a 2.

Based on these rules, obtain:

(a) $P\{\text{Alice wins on the first round given that she rolls a } 3\}$.

Let d_1 be Dilbert's smallest number and d_2 the largest.

Need $d_1 < 3 < d_3$

	1	2	3	4	5	6
1				X	X	X
2				X	X	X
3						
4	X	X				
5	X	X				
6	X	X				

$$\Rightarrow P\{W_1 | A_1 = 3\} = \frac{3(2)(2)}{36} = \frac{12}{36} = \boxed{\frac{1}{3}}$$

(b) $P\{\text{Alice wins on the first round}\}$.

Condition on Alice's 1st roll.

$$P(W_1) = \sum_{k=1}^{6} P(W_1 | A_1 = k) P(A_1 = k) = \frac{1}{6} \sum_{k=1}^{6} P(W_1 | A_1 = k)$$

$$= \frac{1}{6} \frac{1}{36} \left(0 + 1(4)(2) + 3(2)2 + 2(3)2 + 1(4)(2) + 0\right)$$

$$= \frac{40}{216} = \boxed{\frac{5}{27}}$$

(c) $P\{\text{There is a tie on the first round}\}$.

condition on Alice's 1st roll

$$P(T_1) = \sum_{a=1}^{6} P(T_1 \mid A_1 = a) P(A_1 = a) = \frac{1}{6}\frac{1}{36}\left((6+5)6\right) = \boxed{\frac{11}{36}}$$

$$= \frac{66}{216}$$

(d) $P\{\text{Dilbert wins on the first round.}\}$

$$P(D_1) = 1 - \left(P(A_1) + P(T_1)\right) = 1 - \left[\frac{40+66}{216}\right] = \frac{110}{216} = \boxed{\frac{55}{108}}$$

(e) $P\{\text{Alice wins the game}\}$.

$$P(A) = \underbrace{P(A \mid A_1)}_{=1} \underbrace{P(A_1)}_{=\frac{40}{216}} + \underbrace{P(A \mid T_1)}_{=P(A)} \underbrace{P(T_1)}_{=\frac{66}{216}} + \underbrace{P(A \mid D_1)}_{=0} \underbrace{P(D_1)}_{=\frac{110}{216}}$$

memoryless

$$P(A) = \frac{40}{216} + P(A)\frac{66}{216} \Rightarrow P(A) = \frac{\frac{40}{216}}{1 - \frac{66}{216}} = \frac{40}{150} = \boxed{\frac{4}{15}}$$