University of Illinois Summer 2016

ECE 313: Final Exam

Saturday, August 6, 2016, 8-11 a.m.
ECEB 3020

1. [**18 points**] Suppose under hypothesis H_1, X has pdf $f_1(u) = \frac{1}{2} - \frac{1}{4}|u|$ for $u \in (-2, 2)$, but under hypothesis H_0, X is uniformly distributed between $(-1, 1)$.

   (a) Let $\pi_0 = \frac{1}{3}$. Obtain the MAP decision rule.
   Solution: The likelihood ratio is given by $\Lambda(u) = \frac{f_1(u)}{f_0(u)}$. If $u \notin (-1, 1)$, then $f_0(u) = 0$ and hence we chose H_1.
   For $u \in (-1, 1)$, $\Lambda(u) = \frac{\frac{1}{2} - \frac{1}{4}|u|}{\frac{1}{2}} = 1 - \frac{1}{2}|u|$. The MAP rule compares the likelihood ratio to the threshold $\frac{\pi_0}{\pi_1} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$. We have $\Lambda(u) > \frac{1}{2}$, which means $|u| < 1$, so that H_1 is always chosen for $u \in (-1, 1)$.
   Hence the MAP rule is the following: declare H_1 always.

   (b) Obtain $p_{false\ alarm}$ for the MAP rule from part (a).
   Solution: H_1 is always declared under the MAP rule, hence we obtain

   $$p_{false\ alarm} = P\{\text{declare } \ H_1 | H_0\} = 1.$$

   (c) Determine all possible values of π_0 that cause the MAP rule to always choose H_1.
   Solution: Recall that for $u \in (-1, 1)$ we have $\Lambda(u) = 1 - \frac{1}{2}|u| \geq \frac{\pi_0}{1-\pi_0}$. We want this to hold for all $u \in (-1, 1)$, so if it holds for the smallest value of $\Lambda(u)$, it will hold for all. Since $\Lambda(u)$ is decreasing in $|u|$, we need to look at $\Lambda(\pm 1) = \frac{1}{2}$. Solving for π_0 from $\frac{1}{2} \geq \frac{\pi_0}{1-\pi_0}$ we obtain $\pi_0 \in \left[0, \frac{1}{3}\right]$.

2. [**20 points**] Consider a Poisson process of rate λ.

   (a) What is the probability that there are no arrivals in the interval $[0, 2]$?
   Solution: Let N_t be the number of arrivals in $[0, t]$. Then $N_2 \sim Poi(2\lambda)$ because the length if the time interval is 2 time units. Hence,

   $$P(N_2 = 0) = \frac{e^{-2\lambda}(2\lambda)^0}{0!} = e^{-2\lambda}.$$

   (b) What is the probability that there are more than two arrivals in the interval $[0, 2]$?
   Solution:

   $$\begin{aligned} P(N_2 > 2) &= 1 - P(N_2 \leq 2) = 1 - \left(\frac{e^{-2\lambda}(2\lambda)^0}{0!} + \frac{e^{-2\lambda}(2\lambda)^1}{1!} + \frac{e^{-2\lambda}(2\lambda)^2}{2!} \right) \\ &= 1 - e^{-2\lambda}\left(1 + 2\lambda + 2\lambda^2\right). \end{aligned}$$

(c) Given that there are two arrivals during $[0, 2]$, what is the probability that there is one arrival during $[0, 0.25]$?

Solution:

$$\begin{aligned} P(N_{0.25} = 1 | N_2 = 2) &= \frac{P(N_{0.25} = 1, N_2 = 2)}{P(N_2 = 2)} = \frac{P(N_{0.25} = 1, N_2 - N_{0.25} = 1)}{P(N_2 = 2)} \\ &= \frac{P(N_{0.25} = 1)P(N_2 - N_{0.25} = 1)}{P(N_2 = 2)} = \frac{\left(\frac{e^{-0.25\lambda}(0.25\lambda)^1}{1!}\right)\left(\frac{e^{-1.75\lambda}(1.75\lambda)^1}{1!}\right)}{\frac{e^{-2\lambda}(2\lambda)^2}{2!}} \\ &= \frac{7}{32}, \end{aligned}$$

because $N_{0.25} \sim Poi(0.25\lambda)$ and $N_2 - N_{0.25} \sim Poi(1.75\lambda)$.

(d) Obtain the probability that the second arrival occurs after a fixed time $t > 0$.

Solution: In order for the second arrival to occur after time t, there needs to be at most 1 arrival before time t. Hence,

$$P\{N_t \leq 1\} = \frac{e^{-t\lambda}(t\lambda)^0}{0!} + \frac{e^{-t\lambda}(t\lambda)^1}{1!} = e^{-t\lambda}\left(1 + t\lambda\right),$$

because $N_t \sim Poi(t\lambda)$.

3. **[18 points]** Consider a two stage experiment. First, roll a die, with equiprobable sides labeled $1, 2, 3, 4, 4, 5$ (notice that 4 is on two sides of the die and 6 is not on the die). Let X denote the number showing, and then flip a biased coin X times, where tails shows $\frac{3}{4}$ of the time. Let Y be the number of times tails shows.

(a) Obtain $P\{Y = 3 | X = 4\}$.

Solution: Given that $X = 4$, then $Y \sim Binomial\left(4, \frac{3}{4}\right)$ because the coin will be flipped 4 times and the probability of tails in each flip is $\frac{3}{4}$. Therefore,
$P\{Y = 3 | X = 4\} = \binom{4}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^1 = \frac{27}{64}$.

(b) Obtain $P\{Y = 3\}$. Recall that 4 is on two sides of the die and 6 is not on the die.

Solution: Given that $X = k$, then $Y \sim Binomial\left(k, \frac{3}{4}\right)$ because the coin will be flipped k times and the probability of tails in each flip is $\frac{3}{4}$.
Using the law of total probability,
$P\{Y = 3\} = \sum_{k=0}^{5} P\{Y = 3 | X = k\}P\{X = k\} = \sum_{k=3}^{5}\binom{k}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^{k-3} P\{X = k\} = \binom{3}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^0\frac{1}{6} + \binom{4}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^1\frac{2}{6} + \binom{5}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2\frac{1}{6} = \frac{261}{1024}$.

(c) Obtain $P\{X = 4 | Y = 3\}$.

Solution: Using Bayes rule and the result from part (a),

$$P\{X = 4 | Y = 3\} = \frac{P\{Y = 3 | X = 4\}P\{X = 4\}}{P\{Y = 3\}} = \frac{\frac{27}{64}\frac{2}{6}}{\frac{261}{1024}} = \frac{16}{29}$$

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4. **[22 points]** Let X be a binomial random variable with parameters n and p, and let Y be be a binomial random variable with parameters m and q. Assume that X and Y are independent.

(a) Suppose (only for this part) that $n = 9$, $p = \frac{2}{3}$, $m = 16$ and that $q = \frac{1}{4}$. Obtain $E[2X + 3Y - 1]$ and $Var(2X + 3Y - 1)$.
Solution: If $X \sim Binomial(n, p)$, then $E[X] = np$ and $Var(X) = np(1 - p)$. By linearity of expectation

$$E[2X + 3Y - 1] = 2E[X] + 3E[Y] - 1 = 2(9)\frac{2}{3} + 3(16)\frac{1}{4} - 1 = 23.$$

And from scaling of variance and independence of X and Y,

$$Var(2X+3Y-1) = 2^2Var(X)+3^2Var(Y) = 4(9)\frac{2}{3}\left(1-\frac{2}{3}\right)+9(16)\frac{1}{4}\left(1-\frac{1}{4}\right) = 35.$$

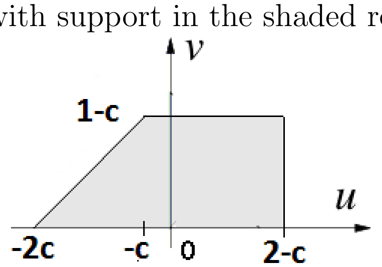
(b) Obtain $Cov(2X, 3Y)$. You can leave your answer in terms of n, p, m and q.
Solution: From the scaling of covariance and independence of X and Y, $Cov(2X, 3Y) = 2(3)Cov(X, Y) = 0$.

(c) Obtain the joint pmf $p_{X,Y}(i, j)$ for all i and j, and express it in terms of p and q.
Solution: If $X \sim Binomial(n, p)$, then $p_X(i) = \binom{n}{i}p^i(1-p)^{n-i}$ for integer $0 \le i \le n$. From the independence of X and Y, $p_{X,Y}(i, j) = p_X(i)p_Y(j) = \binom{n}{i}p^i(1-p)^{n-i}\binom{m}{j}q^j(1-q)^{m-j}$ integer $0 \le i \le n$ and integer $0 \le j \le m$.

(d) Obtain the conditional pmf $p_{X|Y}(i|j)$ for all i and j in terms of n, p, m and q.
Solution: From the independence of X and Y and the fact that $X \sim Binomial(n, p)$, $p_{X|Y}(i|j) = p_X(i) = \binom{n}{i}p^i(1-p)^{n-i}$ for integer $0 \le i \le n$.

(e) Suppose that $n = 6$ but you don't know p. You perform the experiment once and observe that $X^2 = 16$. Obtain the maximum likelihood estimate $\hat{p}_{ML}$.
Solution: The likelihood of observing $X^2 = 16$ is the same as the likelihood of observing $X = 4$, which is $p_X(4) = \binom{6}{4}p^4(1-p)^{6-4}$ because $X \sim Binomial(n, p)$. Maximizing this likelihood (say taking derivatives) yields $\hat{p}_{ML} = \frac{2}{3}$.

5. **[22 points]** Let X and Y be jointly uniform random variables with joint pdf $f_{X,Y}(u, v)$ with support in the shaded region below, where $c \ge 0$ is a constant.



(a) Let $c = \frac{1}{2}$. Obtain the joint pdf $f_{X,Y}(u, v)$ for all points in the $2 - d$ plane.
Solution: For jointly uniform random variables, the joint pdf $f_{X,Y}(u, v)$ is simply inversely proportional to the shaded area

$$A = \frac{\left[-\frac{1}{2} - \left(-2\left(\frac{1}{2}\right)\right)\right]\left(1-\frac{1}{2}\right)}{2} + \left(2 - \frac{1}{2} - \left(-\frac{1}{2}\right)\right)\left(1 - \frac{1}{2}\right) = \frac{9}{8}.$$

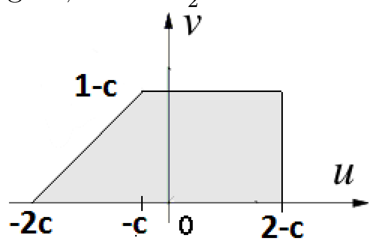
Therefore, $f_{X,Y}(u, v) = \frac{1}{\frac{9}{8}} = \frac{8}{9}$ for $-1 \le u \le \frac{3}{2}$ and $0 \le v \le \min\left\{\frac{1}{2}, u + 1\right\}$.

(b) Again, let $c = \frac{1}{2}$. Obtain the marginal pdf $f_X(u)$ for all u.

Solution: To obtain the marginal of X we need to integrate the joint pdf over all values of Y: $f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)dv$.

$$f_X(u) = \begin{cases} \int_0^{u+1} \frac{8}{9} dv = \frac{8}{9}(u+1) & -1 \le u \le -\frac{1}{2} \\ \int_0^{\frac{1}{2}} \frac{8}{9} dv = \frac{4}{9} & -\frac{1}{2} \le u \le \frac{3}{2} \\ 0 & else \end{cases}$$

(c) Again, let $c = \frac{1}{2}$. Obtain the conditional pdf $f_{Y|X}(v|u)$ for all u and v.



Solution: The conditional pdf $f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(u)}$ when $f_X(u) > 0$. From parts (a) and (b),

$$f_{Y|X}(v|u) = \begin{cases} undefined & u \notin \left[-1, \frac{3}{2}\right] \\ \frac{\frac{8}{9}}{\frac{8}{9}(u+1)} = \frac{1}{u+1} & -1 \le u \le -\frac{1}{2}, 0 \le v \le u+1 \\ \frac{\frac{8}{9}}{\frac{4}{9}} = 2 & -\frac{1}{2} \le u \le \frac{3}{2}, 0 \le v \le \frac{1}{2} \\ 0 & else \end{cases}.$$

(d) Find the value of the constant c such that X and Y are independent.

Solution: A condition for independence of two random variables is that the support of their joint pdf has to be a product set, and this condition is sufficient for independence if the random variables are jointly uniform. Therefore, c must be such that the shaded area is a rectangle: $c = 0$.

6. **[18 points]** Suppose X and Y are jointly Gaussian random variables with $\mu_X = 3$, $\mu_Y = 0$, $\sigma_X^2 = 7$, $\sigma_Y^2 = 9$, and $Cov(X,Y) = 10$. Let $Z = X + Y$. NOTE: you can leave your answers for this problem in terms of the Q function.

(a) Obtain $P(X < 1)$.

Solution: X is a Gaussian random variable with mean 3 and variance 7, hence.

$$P(X < 1) = P\left(\frac{X-3}{\sqrt{7}} < \frac{1-3}{\sqrt{7}}\right) = \Phi\left(-\frac{2}{\sqrt{7}}\right) = Q\left(\frac{2}{\sqrt{7}}\right)$$

(b) Obtain $P(Z < 1)$.

Solution: Observe that Z is a Gaussian random variable with mean $E[Z] = E[X+Y] = E[X] + E[Y] = 3 + 0 = 3$ and variance $Var(Z) = Var(X+Y) = Cov(X+Y, X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) = 7 + 9 + 2(10) = 36$. Hence,

$$P(Z < 1) = P\left(\frac{Z-3}{6} < \frac{1-3}{6}\right) = \Phi\left(-\frac{1}{3}\right) = Q\left(\frac{1}{3}\right)$$

(c) Obtain the best MMSE the linear estimator $\hat{E}[X|Z]$.

Solution: The best MMSE the linear estimator $\hat{E}[X|Z]$ is given by

$$\hat{E}[X|Z] = \frac{Cov(X,Z)}{Var(Z)}(Z - \mu_Z) + \mu_X = \frac{17}{36}(Z-3)+3,$$

because $Cov(X,Z) = Cov(X, X+Y) = Var(X) + Cov(X,Y) = 7 + 10 = 17$.

7. **[18 points]** A store sells two types of phones, model A and model I. An A phone battery drains after X days, where $X \sim Exp(4)$. An I phone battery drains after Y days, where $Y \sim Exp(3)$. The random variables X and Y are independent.

(a) Given that the battery in an A phone has not drained in the the first u days, what is the expected time before its battery drains?

Solution: The expected time before the battery drains, given that the battery has not drained in the the first u days, is

$$E[X|X>u] = E[Z+u] = \frac{1}{\lambda_X} + u = \frac{1}{4} + u,$$

where we used the memoryless property of the exponential random variable X, and $Z \sim Exp(4)$.

(b) Suppose that the I phone is turned on after the A phone's battery drains. Given that the battery in an A phone has not drained in the the first u days, what is the expected total time before both batteries drain?

Solution:

$$E[X+Y|X>u] = E[X|X>u] + E[Y] = \frac{1}{4} + u + \frac{1}{\lambda_Y} = \frac{1}{4} + u + \frac{1}{3} = \frac{7}{12} + u,$$

where we used the linearity of expectation, the memoryless property of the exponential random variable X, $Z \sim Exp(4)$, and the fact the Y is independent of X.

(c) What is the probability that an A phone battery drains before an I phone battery?

Solution: The probability that an A phone battery drains before an I phone battery is given by

$$P\{X<Y\} = \int_0^\infty \int_0^v 4e^{-4u}3e^{-3v}\, du\, dv = \int_0^\infty (1-e^{-4v})3e^{-3v}\, dv = \frac{4}{4+3} = \frac{4}{7}.$$

8. **[22 points]** Let c be a constant and X be a random variable with pdf.

$$f_X(u) = \begin{cases} \frac{1}{2} & u \in [-1, 0), \\ \frac{4}{9}u^2 & u \in [0, c], \\ 0 & \text{else.} \end{cases}.$$

You can leave your answers to this problem in terms of c, except for part (a).

(a) Obtain the value of the constant c in order for $f_X(u)$ to be a valid pdf.

Solution: The pdf has to be non-negative and it has to integrate to one. It is clearly non-negative and

$$1 = \int_{\infty}^{\infty} f_X(u)du = \int_{-1}^{0} \frac{1}{2}du + \int_{0}^{c} \frac{4}{9}u^2 du = \frac{1}{2} + \frac{4}{27}c^3,$$

so that $c = \left(\frac{27}{8}\right)^{1/3} = \frac{3}{2}$.

(b) Determine the CDF $F_X(u)$ for all u. You can leave your answer in terms of the constant c.

Solution: By definition, $F_X(u) = P\{X \leq u\}$. From the support of f_X we can clearly see that $F_X(u) = 0$ if $u < -1$ and $F_X(u) = 1$ if $u > c$.

If $u \in [-1, 0)$ then

$$F_X(u) = \int_{-\infty}^{u} f_X(v)dv = \int_{-1}^{u} \frac{1}{2}du = \frac{u+1}{2}.$$

If $u \in [0, c]$ then

$$F_X(u) = \int_{-\infty}^{u} f_X(v)dv = \int_{-1}^{0} \frac{1}{2}du + \int_{0}^{u} \frac{4}{9}u^2 du = \frac{1}{2} + \frac{4}{27}u^3.$$

So that

$$F_X(u) = \begin{cases} 0 & u < -1, \\ \frac{u+1}{2} & u \in [-1, 0), \\ \frac{1}{2} + \frac{4}{27}u^3 & u \in [0, c], \\ 1 & u > c \end{cases},$$

(c) Obtain $E[X]$ and $E[X^3]$. Recall that $f_X(u) = \begin{cases} \frac{1}{2} & u \in [-1, 0), \\ \frac{4}{9}u^2 & u \in [0, c], \\ 0 & \text{else.} \end{cases}$.

Solution: By definition

$$E[X] = \int_{\infty}^{\infty} u f_X(u)du = \int_{-1}^{0} u\frac{1}{2}du + \int_{0}^{c} u\frac{4}{9}u^2 du = -\frac{1}{4} + \frac{c^4}{9}.$$

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$$E[X^3] = \int_{\infty}^{\infty} u^3 f_X(u)du = \int_{-1}^{0} u^3\frac{1}{2}du + \int_{0}^{c} u^3\frac{4}{9}u^2 du = -\frac{1}{8} + \frac{4}{54}c^6.$$

(d) Let $Y = -\sqrt[3]{X}$. Obtain the CDF $F_Y(v)$ for all v.

Solution: Notice that $X \in [-1, c]$, so that $Y \in [-\sqrt[3]{c}, 1]$. By definition,

$$\begin{aligned} F_Y(v) &= P\{Y \leq v\} = P\{-\sqrt[3]{X} \leq v\} = P\{X \geq -v^3\} = 1 - F_X(-v^3) \\ &= \begin{cases} 0 & v < -\sqrt[3]{c}, \\ 1 - \left(\frac{1}{2} + \frac{4}{27}(-v^3)^3\right) = \frac{1}{2} + \frac{4}{27}v^9 & v \in [-\sqrt[3]{c}, 0), \\ 1 - \left(\frac{-v^3+1}{2}\right) = \frac{1+v^3}{2} & v \in [0, 1], \\ 1 & v > 1 \end{cases} \end{aligned}$$

9. **[12 points]** Suppose that an urn contains g green balls and r red balls. All balls are equally likely to be taken out of the urn.

   (a) Suppose that you grab a total of k balls (no balls are put back). What is the probability of grabbing x green balls?

   Solution: Since the experiment is under a uniform probability distribution, we can solve this problem by counting. We want to count the number of ways to grab x green balls in a total of k balls grabbed. We first select x out of g green balls and then independently select $k-x$ out of r red balls. The number of ways to select x out of g green balls is given by $\binom{g}{x}$, and the number of ways to select select $k-x$ out of r red balls is $\binom{r}{k-x}$. So, there are $\binom{g}{x}\binom{r}{k-x}$ ways of having x green balls among the k grabbed balls. We normalize by the total number of ways to grab k balls, to get the solution:

$$\frac{\binom{g}{x}\binom{r}{k-x}}{\binom{g+r}{k}}.$$

   (b) Now suppose that all k balls are returned to the urn, and this time you grab a total of m balls. Let A be the event that among the set of m balls, exactly 2 green balls are included that were also grabbed the first time. Find $P(A)$.

   Solution: Again, we use counting to solve for the probability. We decompose the counting problem into two steps: we first select the two green balls that are grabbed both times, and then we independently select the remaining $m-2$ balls from the $g+r-x$ balls that were not green balls grabbed the first time. Therefore we have $\binom{x}{2}\binom{g+r-x}{m-2}$, giving us the probability of

$$\frac{\binom{x}{2}\binom{g+r-x}{m-2}}{\binom{g+r}{m}}.$$

10. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

   (a) Let X be a random variable with an even function $f_X(u)$ as its pdf, i.e. $f_X(u) = f_X(-u)$ for all u. Assume $E[|X|^3]$ is finite.

TRUE	FALSE	
☐	☐	$Var(X) = Var(\lvert X\rvert)$.
☐	☐	$E[(1-X)^2] = 1 + Var(X)$
☐	☐	$Var[(1-X)^2] = Var(X)$
☐	☐	$E[(1-X)^3] = 1 + 3Var(X)$.

   Solution: False, True, False, True

(b) Let X and Y be jointly random variables. $*$ denotes convolution.

TRUE	FALSE	
☐	☐	$E[X+Y] = E[X] + E[Y]$ if and only if X and Y are uncorrelated.
☐	☐	$\mathrm{Var}(X+Y) = \sigma_X^2 + \sigma_Y^2$ if and only if X and Y are uncorrelated.
☐	☐	$f_{X+Y}(u) = f_X(u) * f_Y(u)$ if and only if X and Y are uncorrelated.

Solution: False, True, False

(c) Suppose X and Y are jointly Gaussian random variables.

TRUE	FALSE	
☐	☐	If $\hat{E}[Y|X] = 3X + 1$, then $E[Y|X] = 3X + 1$.
☐	☐	If $\hat{E}[Y|X] = 3X + 1$ then $\hat{E}[X|Y] = \frac{1}{3}Y - \frac{1}{3}$.
☐	☐	If $E[Y|X]$ is constant, then $E[X|Y]$ is also constant.

Solution: True, False, True.