University of Illinois Summer 2015

ECE 313: Final Exam

Friday, August 7, 2015
8:00 a.m. — 11:00 a.m.

1. [**24 points**] For this problem, Dilbert flips a fair coin exactly 5 times each minute and Wally rolls multiple fair dice simultaneously multiple times.

   For each part below, **circle the letter of the correct answer and fill in the blanks for your selection.** If you circle more than one answer or if you do not circle an answer, you will receive zero credit.

   NOTE: The problem parts are independent.

   (a) Let M denote the number of heads that occur in the first 2 minutes. What type of random variable is M?

   A. Poisson, $\lambda =$ ____ B. Geometric, $p =$ ____

   C. Binomial, $n =$ ____ $p =$ ____ D. Bernoulli, $p =$ ____

   E. Negative Binomial, $r =$ ____ $p =$ ____ F. Exponential, $\lambda =$ ____

   Solution: There will be exactly 10 coin flips in 2 minutes, and each flip has probability 1/2 of being heads, independently of other flips. Each flip being heads can be modeled using a $Bernoulli(1/2)$ random variable, so the answer is C, $M \sim Binomial(10, 1/2)$.

   (b) Suppose Dilbert flipped his coin for five minutes without observing any heads. Let K denote the number of heads observed in the sixth minute. What type of random variable is K?

   A. Poisson, $\lambda =$ ____ B. Geometric, $p =$ ____

   C. Binomial, $n =$ ____ $p =$ ____ D. Bernoulli, $p =$ ____

   E. Negative Binomial, $r =$ ____ $p =$ ____ F. Exponential, $\lambda =$ ____

   Solution: The coins flips in the sixth minute are independent of the coin flips in the first five minutes. There will be exactly 5 coin flips in the sixth minute, and each flip has probability 1/2 of being heads, independently of other flips. Each flip being heads can be modeled using a $Bernoulli(1/2)$ random variable, so the answer is C, $K \sim Binomial(5, 1/2)$.

(c) Dilbert flips his coin until a heads occurs. Let Q denote the number of coin flips needed. What kind of random variable is Q?

A. Poisson, $\lambda =$ ____

B. Geometric, $p =$ ____

C. Binomial, $n =$ ____ $p =$ ____

D. Bernoulli, $p =$ ____

E. Negative Binomial, $r =$ ____ $p =$ ____

F. Exponential, $\lambda =$ ____

Solution: Each coin flip has probability 1/2 of being heads, independently of other flips, which can be modeled using a $Bernoulli(1/2)$ random variable, so the number of flips needed until a heads occurs is answer is B, $Q \sim Geometric(1/2)$.

(d) Suppose Dilbert flips the coin 7 times. Let H denote the number of heads that occur. What type of random variable is H?

A. Poisson, $\lambda =$ ____

B. Geometric, $p =$ ____

C. Binomial, $n =$ ____ $p =$ ____

D. Bernoulli, $p =$ ____

E. Negative Binomial, $r =$ ____ $p =$ ____

F. Exponential, $\lambda =$ ____

Solution: Each flip has probability 1/2 of being heads, independently of other flips. Each flip being heads can be modeled using a $Bernoulli(1/2)$ random variable, so the answer is C, $H \sim Binomial(7, 1/2)$.

(e) Suppose Dilbert flips the coin 7 times and let H denote the number of heads that occur. After that, Wally rolls H dice simultaneously. Let T denote the number of twos he rolls. The conditional pmf $p_{T|H}$ has which of the following distributions?

A. Poisson, $\lambda =$ ____

B. Geometric, $p =$ ____

C. Binomial, $n =$ ____ $p =$ ____

D. Bernoulli, $p =$ ____

E. Negative Binomial, $r =$ ____ $p =$ ____

F. Exponential, $\lambda =$ ____

Solution: Given that Dilbert got H heads, Wally will roll exactly H dice simultaneosly, and each die rolled has probability 1/6 of being a two, independently of other rolls. Each roll being two can be modeled using a $Bernoulli(1/6)$ random variable, so the answer is C, $T|H \sim Binomial(H, 1/6)$.

(f) Suppose Dilbert flips the coin 7 times and let H denote the number of heads that occur. After that, Wally rolls H dice simultaneously. Let T denote the number of twos he rolls. The random variable Y takes the value one if $T = H$, and zero otherwise. What kind of random variable is Y?

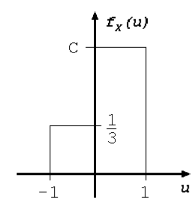
A. Poisson, $\lambda =$ ____ B. Geometric, $p =$ ____

C. Binomial, $n =$ ____ $p =$ ____ D. Bernoulli, $p =$ ____

E. Negative Binomial, $r =$ ____ $p =$ ____ F. Exponential, $\lambda =$ ____

Solution: Y can take the values zero and one.
$P\{Y = 1\} = P\{T = H\} = P\{H \text{ out of } H \text{ coin flips are twos}\} = \left(\frac{1}{6}\right)^H$.
Hence, the answer is D, $Y \sim Bernoulli\left(\left(\frac{1}{6}\right)^H\right)$.

2. [**20 points**] The random variable X has the pdf plotted here.



(a) Obtain the constant c for $f_X(u)$ to be a valid pdf.
Solution: The pdf has to integrate to one, so

$$1 = \int_{-\infty}^{\infty} f_X(u)du = \int_{-1}^{0} \frac{1}{3}du + \int_{0}^{1} cdu = \frac{1}{3} + c.$$

Solving for c gives $c = \frac{2}{3}$.

(b) Obtain $E[6X + 1]$ and $Var(6X + 1)$
Solution: We could use LOTUS to calculate this directly or we can use the scaling of mean and variance. Using the scaling we do

$$E[X] = \int_{-\infty}^{\infty} uf_X(u)du = \int_{-1}^{0} u\frac{1}{3}du + \int_{0}^{1} u\frac{2}{3}du = \frac{1}{6},$$

and

$$E[X^2] = \int_{-\infty}^{\infty} u^2 f_X(u)du = \int_{-1}^{0} u^2\frac{1}{3}du + \int_{0}^{1} u^2\frac{2}{3}du = \frac{1}{3},$$

so that

$$Var(X) = E\left[X^2\right] - \mu_x^2 = \frac{1}{3} - \left(\frac{1}{6}\right)^2 = \frac{11}{36}.$$

Then, by scaling of mean and variance
$E[6X + 1] = 6E[X] + 1 = 2$ and $Var(6X + 1) = 6^2 Var(X) = 11$.
.
If you use LOTUS directly, then $E[6X + 1] = \int_{-\infty}^{\infty}(6u + 1)f_X(u)du = 2$ and $Var(6X + 1) = \int_{-\infty}^{\infty}(6u + 1 - E[6X + 1])^2 f_X(u)du = 11$.

(c) Obtain the CDF of X, $F_X(u)$ for all u.
Solution: By definition,

$$F_X(u) = P\{X \leq u\} = \int_{-\infty}^{u} f_X(u)du = \begin{cases} 0 & u < -1 \\ \int_{-1}^{u} \frac{1}{3}du = \frac{u+1}{3} & u \in [-1,0] \\ F_X(0) + \int_0^u \frac{2}{3}du = \frac{1}{3} + \frac{2}{3}u & u \in (0,1] \\ 1 & u > 1 \end{cases}$$

(d) Obtain $P\{X \geq \frac{1}{2}\}$
Solution: We can write this as

$$P\{X \geq \frac{1}{2}\} = F_X^c\left(\frac{1}{2}\right) = 1 - F_X\left(\frac{1}{2}\right) = 1 - \left(\frac{1}{3} + \frac{2}{3}\frac{1}{2}\right) = \frac{1}{3}.$$

You could also integrate direcctly $P\{X \geq \frac{1}{2}\} = \int_{\frac{1}{2}}^{\infty} f_X(u)du = \frac{1}{3}$.

3. **[22 points]** Let X be an exponential random variable with parameter 2 and let Y be an exponential random variable with parameter 2, independent of X.

(a) Obtain the joint pdf of X and Y, $f_{X,Y}(u,v)$ for all u and v.
Solution: The joint pdf of independent random variable can be obtained by

$$f_{X,Y}(u,v) = f_X(u)f_Y(v) = \begin{cases} 2e^{-2u}2e^{-2v} = 4e^{-2(u+v)} & u,v \geq 0 \\ 0 & \text{else} \end{cases}$$

(b) Obtain the covariance between X and Y.
Solution: Independent random variables are uncorrelated, so $Cov(X,Y) = 0$.

(c) Obtain the covariance between $2X+1$ and $3X-2$.
Solution: Using the bilinearity of covariance $Cov(2X+1, 3X-2) = 2(3)Cov(X,X) = 6\sigma_X^2 = 6\frac{1}{\lambda^2} = 6\frac{1}{2^2} = \frac{3}{2}$.

(d) Let $Z = \frac{2Y}{X+Y}$. Obtain the pdf of Z, $f_Z(c)$ for all c.
Solution: The support of $f_{X,Y}$ is the first quadrant, so the range of Z is $[0,2]$. First obtain the CDF of Z over $[0,2]$:

$$\begin{aligned} F_Z(c) &= P\{Z \leq c\} = P\left\{\frac{2Y}{X+Y} \leq c\right\} = P\left\{Y \leq \frac{c}{2-c}X\right\} \\ &= \int_{-\infty}^{\infty}\int_0^{\frac{c}{2-c}u} f_{X,Y}(u,v)dvdu = \int_0^{\infty}\int_0^{\frac{c}{2-c}u} 4e^{-2(u+v)}dvdu = \frac{c}{2}. \end{aligned}$$

We can see that this is a $Uniform(0,2)$ random variable, so $f_Z(c) = \frac{1}{2}$ for $c \in [0,2]$ and zero else.
To get the pdf from the CDF you could also differentiate $F_Z(c)$ with respect to c.

4. **[20 points]** Calvin is eating a box of 10 raisins by grabbing one at a time, tossing it in the air, and catching it with his mouth. If he doesn't catch it, then Hobbes eats it instead. Calvin catches the raisin in each attempt with probability $p \in (0,1)$, independently of any other attempts. Let $X_i = 1$ if Calvin catches the raisin in his i-th attempt, and $X_i = 0$ if he doesn't. Let X be the total number of raisins he catches.

(a) Obtain $P\{X=3|X_1=1, X_{10}=0\}$.
Solution: The tosses are independent, so if there are 3 successes total, and the first and last tosses are success and failure, respectively, then the other 2 succeses must occur in the remaining 8 tosses. This is a $Binomial(8, p)$ random variable, so $P\{X=3|X_1=1, X_{10}=0\} = \binom{8}{2}p^2(1-p)^6$.
We could also do this directly using the definition of contitional probability:

$$P\{X=3 \mid X_1=1, X_{10}=0\} = \frac{P\{X=3,X_1=1,X_{10}=0\}}{P\{X_1=1,X_{10}=0\}} = \frac{P\{Y=2,X_1=1,X_{10}=0\}}{P\{X_1=1,X_{10}=0\}}$$
$$= \frac{P\{Y=2\}P\{X_1=1,X_{10}=0\}}{P\{X_1=1,X_{10}=0\}} = P\{Y=2\} = \binom{8}{2}p^2(1-p)^6$$

where $Y \sim Binomial(8, p)$ is the number of successes in tosses 2 through 8.

(b) Obtain the conditional pmf of X_1 given that $X_{10}=0$ and $X=3$, that is, obtain $P\{X_1=k|X_{10}=0, X=3\}$, for all k.
Solution: Using the definition of contitional probability, for $k \in \{0,1\}$,

$$P\{X_1=k|X_{10}=0, X=3\} = \frac{P\{X_1=k,X_{10}=0,X=3\}}{P\{X_{10}=0,X=3\}} = \frac{P\{X_1=k,X_{10}=0,Y=3-k\}}{P\{X_{10}=0,Z=3\}}$$
$$= \frac{P\{X_1=k\}P\{X_{10}=0\}P\{Y=3-k\}}{P\{X_{10}=0\}P\{Z=3\}} = P\{X_1=k\}\frac{\binom{8}{3-k}p^{3-k}(1-p)^{8-(3-k)}}{\binom{9}{3}p^3(1-p)^6}$$
$$= \begin{cases} \frac{2}{3} & k=0 \\ \frac{1}{3} & k=1 \end{cases}$$

5. **[18 points]** Suppose that the eleven players in your soccer team go out to dinner and a movie. The team consists of one goalie, four defenders, four midfielders and two forwards.

(a) At the movie theater, the eleven of you seat next to each other on a single row. What is the probability that the four defenders are seated next to each other?
Solution: There are 11! ways to seat 11 players in a single row, which is the size of the sample space. Consider the four defenders as a single unit, so there are now 8 'people' to sit in order, which can be done in 8! ways. For each of those seatings, the four defenders can seat in 4! ways among themselves. Hence,

$$P\{\text{four defenders are seated next to each other}\} = \frac{8!4!}{11!} = \frac{2}{165}.$$

(b) After the movie, the eleven of you all go out to dinner and seat at a circular table. What is the probability that the four defenders are seated next to each other now? Note: rotated seatings are considered the same, but mirror images are not.
Solution: Rotated seatings are considered the same, so define the seatings relative to a specific player. Then, there are 10! ways to seat remaining 10 players, which is the size of the sample space. Consider again the four defenders as a single unit, so there are again 8 'people' to sit in order. Choose a player that is not a defender to be the point of reference, so there are now 7 'people' to sit in order relative to this chosen player, which can be done in 7! ways. For each of those seatings, the four defenders can seat in 4! ways among themselves. Hence,

$$P\{\text{four defenders are seated next to each other}\} = \frac{7!4!}{10!} = \frac{1}{30}.$$

(c) After dinner, the eleven of you all go get icecream and seat at a circular table. What is the probability that the two forwards are not seated next to each other in this circular table? Note: rotated seatings are considered the same, but mirror images are not.

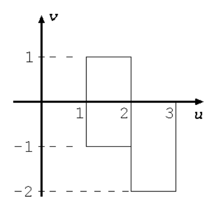
Solution: The sample space is again 10!. Following the same approach as in part (b) but with two forwards instead of four defenders, we find that

$$P\{\text{two forwards are seated next to each other}\} = \frac{9!2!}{10!} = \frac{1}{5}.$$

Using the complement

$$P\{\text{two forwards are not seated next to each other}\} = 1 - \frac{1}{5} = \frac{4}{5}.$$

6. **[24 points]** Let X and Y be jointy continuous random variables with joint pdf $f_{X,Y}(u,v) = ce^{-u}$ on the support set plotted below. Note: you can leave all of your answers for this problem in terms of c, without substituting its value, $c = \frac{1}{2(e^{-1}-e^{-3})}$.



(a) Obtain the marginal distribution of X, $f_X(u)$ for all u.

Solution: By definition,

$$f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)dv = \begin{cases} \int_{-1}^{1} ce^{-u}dv = 2ce^{-u} & u \in [1,2] \\ \int_{-2}^{0} ce^{-u}dv = 2ce^{-u} & u \in [2,3] \\ 0 & \text{else} \end{cases}$$

(b) Obtain the conditional distribution of Y given X, $f_{Y|X}(v|u)$ for all $(u,v) \in \mathbb{R}^2$.

Solution:

$$f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(u)} = \begin{cases} \text{undefined} & u \notin [1,3] \\ \frac{ce^{-u}}{2ce^{-u}} = \frac{1}{2} & u \in [1,2], v \in [-1,1] \Longrightarrow Uniform(-1,1) \\ \frac{ce^{-u}}{2ce^{-u}} = \frac{1}{2} & u \in [2,3], v \in [-2,0] \Longrightarrow Uniform(-2,0) \\ 0 & \text{else} \end{cases}$$

(c) Obtain the best unconstrained estimator of Y from X, $\hat{Y} = g^*(X)$.

Solution: The best unconstrained estimator is the conditional mean:

$$\hat{Y} = g^*(X) = E[Y|X] = \begin{cases} 0 & X \in [1,2] \\ -1 & X \in [2,3] \end{cases}$$

(d) Obtain $P\left\{Y \leq -\frac{1}{2}\right\}$.
Solution:

$$P\left\{Y \leq -\tfrac{1}{2}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{-\frac{1}{2}} f_{X,Y}(u,v)dvdu = \int_1^2\int_{-1}^{-\frac{1}{2}} 2ce^{-u}dvdu + \int_2^3\int_{-2}^{-\frac{1}{2}} 2ce^{-u}dvdu$$
$$= c\left(\frac{e^{-1}+2e^{-2}-3e^{-3}}{2}\right)$$

7. **[24 points]** Let X and Y be jointly Gaussian random variables. The conditional distribution of Y given X is given by $f_{Y|X}(v|u) = \frac{1}{\sqrt{10\pi}}\exp\left(-\frac{(v-u-6)^2}{10}\right)$ for all $(u,v) \in \mathbb{R}^2$. It is known that the best constant estimator of Y from X is $\delta^* = 5$ and its corresponding minimum mean squared error $\text{MMSE}_{\delta^*} = 9$.

   (a) Obtain the marginal pdf of Y, $f_Y(v)$ for all v.
   Solution: It is know that $\delta^* = E[Y] = 5$ and $\text{MMSE}_{\delta^*} = Var(Y) = 9$. Also, if X and Y are jointly Gaussian, then Y is Gaussian, so $Y \sim N(5, 9)$.

   (b) Obtain the best linear estimate of Y given that $X = 4$, $\hat{E}[Y|X=4]$.
   Solution: It is known that $\hat{E}[Y|X] = E[Y|X]$, and the conditional distribution of Y given X is Gaussian with mean $\hat{E}[Y|X]$, as observed in the given $f_{Y|X}(v|u)$, whose mean is $u+6$. So $\hat{E}[Y|X=4] = E[Y|X=4] = 4+6 = 10$

   (c) Obtain the covariance between X and Y, $Cov(X,Y)$.
   Solution: The slope of the linear estimator $\hat{E}[Y|X] = X + 6$ is $\frac{Cov(X,Y)}{Var(X)} = 1$. The conditional distribution of Y given X is Gaussian with variance given by $5 = MMSE_{\hat{E}[Y|X]} = Var(Y) - \frac{Cov^2(X,Y)}{Var(X)} = Var(Y) - \frac{Cov(X,Y)}{Var(X)}Cov(X,Y)$. Subsituting $Var(Y) = 9$ and $\frac{Cov(X,Y)}{Var(X)} = 1$ we can solve for $Cov(X,Y) = 4$.

   (d) Obtain the marginal pdf of X, $f_X(u)$ for all u.
   Solution: From part(c), $\frac{Cov(X,Y)}{Var(X)} = 1$, so that $Var(X) = 4$. The y-intercept of the linear estimator $\hat{E}[Y|X] = X + 6$ is given by $\mu_Y - \frac{Cov(X,Y)}{Var(X)}\mu_X = 6$. Subsituting $\mu_Y = 5$ and $\frac{Cov(X,Y)}{Var(X)} = 1$ we can solve for $\mu_X = -1$. If X and Y are jointly Gaussian, then X is Gaussian, so $X \sim N(-1, 4)$.

8. **[18 points]** The two parts of this problem are unrelated.

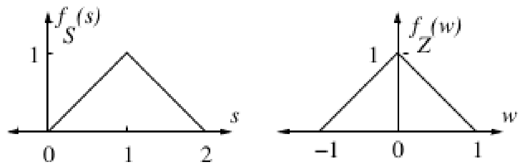
   (a) Let X be uniformly distributed on $(0,1)$, and if $u \in (0,1)$, then $f_{Y|X}(v|u) = \frac{1}{1-u}$ for $0 \leq v < 1-u$, and zero else. Let $S = X + Y$. Obtain the pdf of S, $f_S(c)$.
   Solution: The range of S is $[0,1]$. Recall that the pdf of a sum of random variables can be obtained from

$$f_S(c) = \int_{-\infty}^{\infty} f_{X,Y}(s, c-s)ds = \int_{-\infty}^{\infty} f_{Y|X}(c-s|s)f_X(s)ds = \int_0^c \frac{1}{1-s}(1)ds = -\ln(1-c)$$

   for $c \in [0,1)$.

   (b) In this case, X and Y are independent random variables, each of which is uniformly distributed on $(0,1)$, and let Z be a random variable such that $X + Y + Z = 1$. Clearly sketch the pdf of Z, $f_Z(w)$.

Solution: Rewrite $Z = 1-(X+Y)$, and let $S = X+Y$. Then, the pdf of a sum of independent random variables can be obtained from $f_S(s) = f_X(s) * f_Y(s) =$. As see in class, the convolution of the pdf of two $Uniform(0,1)$ gives the triangular pdf plotted below. The pdf of Z is then simply obtained by a linear scaling of this triangular pdf, $f_Z(w) = \frac{1}{|-1|} f_S\left(\frac{w-1}{-1}\right) = f_S(1-w)$ and is plotted below too.



9. **[30 points]** (3 points per answer)
   In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

   (a) A, B, C are three events such that $0 < P(A) < 1$, $0 < P(B) < 1$ and $0 < P(C) < 1$.

TRUE	FALSE	
☐	☐	$P(A\|B) + P(A^c\|B) = 1$.
☐	☐	$P(A\|B)P(B) + P(A^c\|B)P(B) = P(A)$.
☐	☐	$P(A\|B) = P(B\|A)$, then $P(A) = P(B)$.

   Solution: TRUE, FALSE, FALSE.

   (b) Consider a binary hypothesis testing problem where the prior probability of hypothesis H_0 is π_0 and the prior probability of hypothesis H_1 is π_1. Denote the probabilities of false alarm and missed detection for the ML decision rule by P_{FA}^{ML} and P_{MD}^{ML}, respectively. Similarly, denote the probabilities of false alarm and missed detection for the MAP decision rule by P_{FA}^{MAP} and P_{MD}^{MAP}, respectively.

TRUE	FALSE	
☐	☐	$P_{FA}^{ML} + P_{MD}^{ML} = 1$.
☐	☐	$P_{FA}^{MAP} \leq P_{FA}^{ML}$.
☐	☐	$P_{FA}^{ML} \cdot \pi_0 + P_{MD}^{ML} \cdot \pi_1 \geq P_{FA}^{MAP} \cdot \pi_0 + P_{MD}^{MAP} \cdot \pi_1$.
☐	☐	If $\pi_0 = 0.5$ then $P_{MD}^{ML} = P_{MD}^{MAP}$.

   Solution: FALSE,FALSE,TRUE,TRUE.
   These are conditional probabilities.
   The prior probability π_0 could be very small leading to this scenario.
   An important property of the MAP (MEP) detection rule.
   When the prior probabilities are equal, the MAP and ML rule decision rules coincide.

(c) Suppose X, Y, Z are independent, Bernoulli($\frac{1}{2}$) random variables.

TRUE	FALSE	
☐	☐	$X + Y$ is independent of $X - Y$.
☐	☐	$\text{Cov}(X + 2Y, 2X - Y) = 0$.
☐	☐	$\text{Cov}(XY, XZ) = \frac{1}{8}$

Solution: FALSE, TRUE, FALSE.