University of Illinois Summer 2015

ECE 313: Hour Exam I

Thursday, July 9, 2015
4:00 p.m. — 5:15 p.m.
1013 ECEB

1. **[15 points]** The two parts of this problem are unrelated.

   (a) Suppose that you buy a variety pack of gum, which has 10 pieces of gum. You know 4 of those pieces are cherry flavored. If you take out 5 pieces of gum at random, what is the probability that you get 2 cherry flavored pieces?

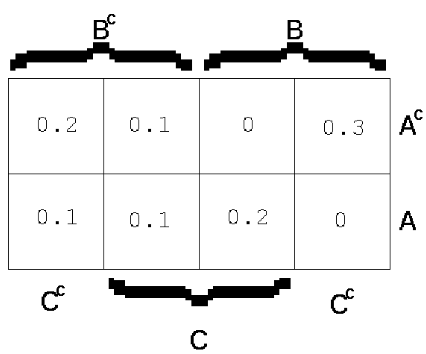
   Solution: This is an application of the hypergeometric distribution we saw in class with balls replaced by pieces of gum. Let A be the event that you get 2 cherry flavored pieces, then

   $$P(A) = \frac{|A|}{|\Omega|} = \frac{\binom{4}{2}\binom{6}{3}}{\binom{10}{5}} = \frac{10}{21}$$

   . The sample space, Ω, is the set of all possible subsets of 5 pieces of gum out of the 10 available pieces, which has cardinality $|\Omega| = \binom{10}{5}$. Event A, getting 2 cherry flavored pieces, can occur in $\binom{4}{2}\binom{6}{3}$ ways because one can choose 2 of the 4 cherry pieces, and for each one of those choices, one can choose 3 out of the 6 non-cherry flavored pieces.

   (b) Suppose A, B, and C are events for a probability experiment such that A and B are independent, $P(A) = P(C) = 0.4$, $P(B) = 0.5$ $P(AC) = 0.3$, $P(BC) = 0.2$ and $P(ABC) = 0.2$. Obtain $P(AB^cC^c)$, $P(A^cBC^c)$, and $P(A^cB^cC^c)$.

   Solution: This can be easily done using a Karnaugh map. Start by using the fact that $P(ABC) = 0.2$ and that $P(AC) = 0.3$ to obtain $P(AB^cC) = 0.1$. Then, from $P(BC) = 0.2$ we obtain $P(A^cBC) = 0$. From $P(C) = 0.4$ we get obtain $P(A^cB^cC) = 0.1$. The independence of A and B implies that $P(AB) = P(A)P(B) = 0.4(0.5) = 0.2$, which yields $P(ABC^c) = 0$.
   From $P(A) = 0.4$, we get $P(AB^cC^c) = 0.1$.
   From $P(B) = 0.5$, we get $P(A^cBC^c) = 0.3$.
   From $P(B^c) = 0.5$, we get $P(A^cB^cC^c) = 0.2$.



2. **[25 points]** Consider rolling a fair die and flipping a fair coin. Let X be a random variable defined as follows. If the coin shows heads, then X is equal to the number showing on the die. If the coin shows tails, then X is equal to the number showing on the die plus one.

   (a) Obtain the pmf of X.

   Solution: The sample space here is $\Omega = \{(n, s) : n \in \{1, 2, 3, 4, 5, 6\}, s \in \{H, T\}\}$, with all twelve outcomes of the experiment being equally likely. Notice that X can take values in the set $\{1, 2, \ldots, 7\}$. Event $\{X = 1\}$ only occurs if the coin shows heads and the number showing on the die is one. Event $\{X = 7\}$ only occurs if the coin shows tails and the number showing on the die is six. All other events $\{X = k\}$ can occur with two possible outcomes, if the coin shows heads and the number showing on the die is k, or if the coin shows tails and the number showing on the die is $k - 1$. So, the pmf of X is given by

$$p_X(k) = \begin{cases} \frac{1}{12} & k \in \{1, 7\} \\ \frac{1}{6} & k \in \{2, 3, 4, 5, 6\} \end{cases}$$

(b) Obtain $E[X]$ and $\text{Var}(X)$.
Solution: Notice that the pmf is symmetric around $k = 4$ so $E[X] = 4$.
We could also calculate the mean by using the definition:

$$E[X] = \sum_{u_i} u_i p_X(u_i) = \sum_{k=1}^{7} k p_X(k) = (1)\frac{1}{12} + (2)\frac{1}{6} + (3)\frac{1}{6} + \cdots + (6)\frac{1}{6} + (7)\frac{1}{12} = \frac{48}{12} = 4$$

$$\begin{aligned} \text{Var}(X) &= E\left[X^2\right] - (E[X])^2 = \textstyle\sum_{u_i} u_i^2 p_X(u_i) \ - \ 4^2 = \sum_{k=1}^{7} k^2 p_X(k) \ - \ 16 \\ &= (1)^2\tfrac{1}{12} + (2)^2\tfrac{1}{6} + (3)^2\tfrac{1}{6} + \cdots + (6)^2\tfrac{1}{6} + (7)^2\tfrac{1}{12} \ - \ 16 = \tfrac{115}{6} - 16 = \tfrac{19}{6} \end{aligned}$$

(c) Obtain $E\left[\frac{X+1}{2}\right]$ and $\text{Var}\left(\frac{X+1}{2}\right)$.
Solution: Using the scaling of mean and variance:

$$E\left[\frac{X+1}{2}\right] = E\left[\frac{1}{2}X + \frac{1}{2}\right] = \frac{1}{2}E[X] + \frac{1}{2} = \frac{1}{2}4 + \frac{1}{2} = \frac{5}{2}$$

.

$$\text{Var}\left(\frac{X+1}{2}\right) = \text{Var}\left(\frac{1}{2}X + \frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 \text{Var}(X) = \frac{1}{4}\frac{19}{6} = \frac{19}{24}$$

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3. **[15 points]** Let X denote a discrete random variable that takes on even integer values $0, 2, 4, \ldots, n$. and zero otherwise.

(a) Let the pmf of X be given by $p_X(k) = \frac{3(2^k)}{4(2^n)-1}$ for even integer values $k \in \{0, 2, 4, \ldots, n\}$, where the value of n is unknown. Find the maximum-likelihood estimate $\hat{n}_{ML}$ from the observation that $X = 10$ on a trial of the experiment.
Solution: The maximum-likelihood estimate $\hat{n}_{ML}$ is the value of n that maximizes the likelihood of the observation $X = 10$: $p_X(10) = \frac{3(2^{10})}{4(2^n)-1}$.
Notice that as n increases, the denominator increases and hence $p_X(10)$ decreases. So we need to choose the smallest possible even integer $n \geq 0$, which is $n = 10$ because if we choose $n < 10$ then we could have not observed $X = 10$. Therefore, $\hat{n}_{ML}$=10.

(b) Now, let $p_X(k) = a$ for even integer values $k \in \{0, 2, 4, \ldots, n\}$, and zero otherwise. Find the constant a that makes this a valid pmf and find its mean.
Solution: For it to be a valid pmf we need:

$$1 = \sum_{u_i} p_X(u_i) = \sum_{k=0}^{n/2} p_X(2k) = \sum_{k=0}^{n/2} a = a\left(\frac{n}{2} + 1\right) \Rightarrow a = \frac{1}{\frac{n}{2} + 1} = \frac{2}{n+2}.$$

The mean can be obtained by inspection because all values of X are equally likely, so the mean is the midpoint, hence $\mu_X = E[X] = \frac{n}{2}$.
The mean can also be obtained using the definition

$$E[X] = \sum_{u_i} p_X(u_i) = \sum_{k=0}^{n/2} (2k) p_X(2k) = \sum_{k=0}^{n/2} 2ka = 2a \sum_{k=0}^{n/2} k = 2a \frac{\frac{n}{2}\left(\frac{n}{2} + 1\right)}{2} = a\frac{n(n+2)}{4} = \frac{n}{2}.$$

4. **[20 points]** Consider a Bernoulli process $X = (X_1, X_2, \ldots)$ with $P\{X_1 = 1\} = p \in (0, 1)$. A sample path representing its cumulative number of successes is given by $C = (C_0, C_1, \ldots) = (0, ?, 2, ?, 2, ?, 3, 4, 4, \ldots)$, where ? indicates that the value is unknown. The corresponding sample path representing the number of trials between success is given by $L = (L_1, L_2, \ldots) = (1, 1, ?, 2, 4, \ldots)$, and the corresponding sample path representing the cumulative number of trials until the j-th success is given by $S = (S_0, S_1, \ldots) = (0, 1, 2, 5, ?, 11, \ldots)$.

   (a) Find the values of X_7, L_3, S_4, and C_5.
   Solution: Recall that $X_7 = C_7 - C_6 = 4 - 3 = 1$.
   $L_3 = S_3 - S_2 = 5 - 2 = 3$.
   $S_4 = L_1 + L_2 + \cdots + L_4 = 1 + 1 + 3 + 2 = 7$.
   $C_4 \le C_5 \le C_6$, so $2 \le C_5 \le 3$. From S_3 we can see that there was a success at X_5, hence $C_5 = 3$.

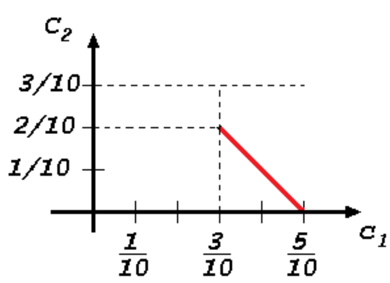
   (b) Determine $P\{X_6 = 0\}$, $P\{C_5 = 2\}$, $P\{L_3 = 3\}$ and $P\{S_3 = 5\}$, in terms of p.
   Solution: Recall that $X_k \sim \mathit{Bernoulli}(p)$, hence $P\{X_6 = 0\} = 1 - p$.
   $C_k \sim \mathit{Binomial}(k, p)$, hence $P\{C_5 = 2\} = \binom{5}{2} p^2 (1-p)^3$.
   $L_j \sim \mathit{Geometric}(p)$, hence $P\{L_3 = 3\} = (1-p)^2 p$.
   $S_j \sim \mathit{Negative\ Binomial}(j, p)$, hence $P\{S_3 = 5\} = \binom{4}{2} p^3 (1-p)^2$.

5. **[25 points]** An analog-to-digital converter digitizes signals to 5 values: $-2, -1, 0, 1, 2$. The pmfs of digitized signals A and B are given by

k	-2	-1	0	1	2
$p_A(k)$	$\frac{1}{8}$	c_1	$\frac{2}{8}$	c_2	$\frac{1}{8}$

k	-2	-1	0	1	2
$p_B(k)$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{10}$

   (a) If an ML rule is used, sketch the region in the (c_1, c_2) plane where signal A would be determined to have been digitized if we observe a value of $X = -1$ but signal B would be determined to have been digitized if we observe a value of $X = 1$.
   Solution: If a ML rule was used, to determine that signal A had been digitized if we observe a value of $X = -1$, we'd need $p_A(-1) > p_B(-1)$ so that $c_1 > \frac{3}{10}$. To determine that signal B had been digitized if we observe a value of $X = 1$, we'd need $p_A(1) < p_B(1)$ so that $c_2 < \frac{3}{10}$. However, we must make sure that the pmf of A is valid, so that $1 = \sum_{u_i} p_A(u_i) = \frac{1}{8} + c_1 + \frac{2}{8} + c_2 + \frac{1}{8} = \frac{1+2(c_1+c_2)}{2}$, which gives the constraint that $c_2 = \frac{1}{2} - c_1$. The corresponding region is the line sketched in the figure below.



   (b) Suppose now that $c_1 = c_2 = \frac{2}{8}$, determine the value(s) of $P\{\text{signal } B\}$ in order to always determine that signal B was digitized under the MAP rule.
   Solution: Under the MAP rule, signal B is always determined to have been digitized if $\Lambda(k) = \frac{p_B(k)}{p_A(k)} \ge \frac{P\{\text{signal } A\}}{P\{\text{signal } B\}}$ for all k. With these values of c_1 and c_2, the minimum of the likelohood ratio $\Lambda(k) = \frac{p_B(k)}{p_A(k)}$ is $\frac{4}{10} = \frac{2}{5}$. Recall that $P\{\text{signal } A\} + P\{\text{signal } B\} = 1$. Hence,

$$\frac{2}{5} > \frac{1 - P\{\text{signal } B\}}{P\{\text{signal } B\}},$$

which yields $\frac{5}{7} \leq P\{\text{signal } B\}$. We know that $P\{\text{signal } B\} \leq 1$ because it is a probability so $\frac{5}{7} \leq P\{\text{signal } B\} \leq 1$.