University of Illinois Summer 2015

ECE 313: Hour Exam I

Thursday, July 9, 2015
4:00 p.m. — 5:15 p.m.
1013 ECEB

Name: (in BLOCK CAPITALS) ____________________________________

NetID: ____________________________________

Signature: ____________________________________

Instructions

This exam is closed book and closed notes except that one 8.5”×11” sheet of notes is permitted: both sides may be used. No electronic equipment (cell phones, etc.) allowed.

The exam consists of six problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading

1. 15 points ____________
2. 25 points ____________
3. 15 points ____________
4. 20 points ____________
5. 25 points ____________

Total (100 points) ____________

1. **[15 points]** The two parts of this problem are unrelated.

   (a) Suppose that you buy a variety pack of gum, which has 10 pieces of gum. You know 4 of those pieces are cherry flavored. If you take out 5 pieces of gum at random, what is the probability that you get 2 cherry flavored pieces?

   (b) Suppose A, B, and C are events for a probability experiment such that A and B are independent, $P(A) = P(C) = 0.4$, $P(B) = 0.5$ $P(AC) = 0.3$, $P(BC) = 0.2$ and $P(ABC) = 0.2$. Obtain $P(AB^cC^c)$, $P(A^cBC^c)$, and $P(A^cB^cC^c)$.

2. **[25 points]** Consider rolling a fair die and flipping a fair coin. Let X be a random variable defined as follows. If the coin shows heads, then X is equal to the number showing on the die. If the coin shows tails, then X is equal to the number showing on the die plus one.

   (a) Obtain the pmf of X.

   (b) Obtain $E[X]$ and $\text{Var}(X)$.

   (c) Obtain $E\left[\frac{X+1}{2}\right]$ and $\text{Var}\left(\frac{X+1}{2}\right)$.

3. **[15 points]** Let X denote a discrete random variable that takes on even integer values $0, 2, 4, \ldots, n$. and zero otherwise.

(a) Let the pmf of X be given by $p_X(k) = \frac{3(2^k)}{4(2^n)-1}$ for even integer values $k \in \{0, 2, 4, \ldots, n\}$, where the value of n is unknown. Find the maximum-likelihood estimate $\hat{n}_{ML}$ from the observation that $X = 10$ on a trial of the experiment.

(b) Now, let $p_X(k) = a$ for even integer values $k \in \{0, 2, 4, \ldots, n\}$, and zero otherwise. Find the constant a that makes this a valid pmf and find its mean.

4. **[20 points]** Consider a Bernoulli process $X = (X_1, X_2, \ldots)$ with $P\{X_1 = 1\} = p \in (0, 1)$. A sample path representing its cumulative number of successes is given by $C = (C_0, C_1, \ldots) = (0, ?, 2, ?, 2, ?, 3, 4, 4, \ldots)$, where ? indicates that the value is unknown. The corresponding sample path representing the number of trials between success is given by $L = (L_1, L_2, \ldots) = (1, 1, ?, 2, 4, \ldots)$, and the corresponding sample path representing the cumulative number of trials until the j-th success is given by $S = (S_0, S_1, \ldots) = (0, 1, 2, 5, ?, 11, \ldots)$.

(a) Find the values of X_7, L_3, S_4, and C_5.

(b) Determine $P\{X_6 = 0\}$, $P\{C_5 = 2\}$, $P\{L_3 = 3\}$ and $P\{S_3 = 5\}$, in terms of p.

5. **[25 points]** An analog-to-digital converter digitizes signals to 5 values: $-2, -1, 0, 1, 2$. The pmfs of digitized signals A and B are given by

k	-2	-1	0	1	2
$p_A(k)$	$\frac{1}{8}$	c_1	$\frac{2}{8}$	c_2	$\frac{1}{8}$

k	-2	-1	0	1	2
$p_B(k)$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{10}$

(a) If an ML rule is used, sketch the region in the (c_1, c_2) plane where signal A would be determined to have been digitized if we observe a value of $X = -1$ but signal B would be determined to have been digitized if we observe a value of $X = 1$.

(b) Suppose now that $c_1 = c_2 = \frac{2}{8}$, determine the value(s) of $P\{\text{signal } B\}$ in order to always determine that signal B was digitized under the MAP rule.