University of Illinois Summer 2014

ECE 313: Exam I

Monday, July 7, 2014
7:00 p.m. — 8:15 p.m.
269 Everitt Lab

1. (a) Let L denote the restaurant quality level. Then $\mathbb{P}[L=\ell]=\frac{1}{5}$ for $\ell=1,\ldots,5$. Let Y_ℓ denote the Poisson distribution with mean 10ℓ for $\ell=1,\ldots,5$. Let X denote the number of people dining in the restaurant. According to the total probability formula,

$$\mathbb{P}[X=40]=\sum_{\ell=1}^{5}\mathbb{P}[X=40|L=\ell]\mathbb{P}[L=\ell]=\frac{1}{5}\sum_{\ell=1}^{5}\mathbb{P}[Y_\ell=40]=\frac{1}{5}\sum_{\ell=1}^{5}\frac{(10\ell)^{40}}{40!}e^{-10\ell}.$$

(b) By the Bayes formula,

$$\mathbb{P}[L\geq 3|X=40]=\frac{\sum_{\ell=3}^{5}\mathbb{P}[X=40|L=\ell]\mathbb{P}[L=\ell]}{\mathbb{P}[X=40]}=\frac{\sum_{\ell=3}^{5}\frac{(10\ell)^{40}}{40!}e^{-10\ell}}{\sum_{\ell=1}^{5}\frac{(10\ell)^{40}}{40!}e^{-10\ell}}.$$

2. (a) Define the likelihood ratio $\Lambda(k)=\frac{\mathbb{P}[X=k|H_1]}{\mathbb{P}[X=k|H_0]}$. Then $\Lambda(0)=\frac{1-q}{1-p}=\frac{2}{3}$ and $\Lambda(1)=\frac{q}{p}=\frac{3}{2}$. Hence, the ML decides H_0 when $X=0$ and decides H_1 when $X=1$.

(b) By definition,

$$p_{\text{false alarm}}=\mathbb{P}[\text{ML says } H_1|H_0]=\mathbb{P}[X=1|H_0]=p=0.4$$
$$p_{\text{miss}}=\mathbb{P}[\text{ML says } H_0|H_1]=\mathbb{P}[X=0|H_0]=1-q=0.4.$$

(c) The MAP decides H_0 if $\Lambda(k)<\frac{P(H_0)}{P(H_1)}$ and decides H_1 if $\Lambda(k)>\frac{P(H_0)}{P(H_1)}$. In this problem, $\frac{P(H_0)}{P(H_1)}=\frac{1}{3}$ and therefore the MAP always say H_1 no matter what values the observation of X takes.

(d) By definition

$$p_e=P(H_0)p_{\text{false alarm}}+P(H_1)p_{\text{miss}}=\frac{1}{4}\times 1+\frac{3}{4}\times 0=\frac{1}{4}.$$

(e) Note that $p_{\text{false alarm}}$ for the MAP rule is zero means that the MAP always say H_0, which implies that $\frac{P(H_0)}{P(H_1)}>\frac{3}{2}$ (here is the inequality is strict, because we assume the tie is broken in favor of H_1), and thus $0.6<P(H_0)\leq 1$.

3. (a) The solution is $\frac{d}{m}$. We made an error only if $F(x)$ is not identical to $G(x)$ and n is a root of $F(x)-G(x)$. Since there are at most d roots of $F(x)-G(x)$ and n is uniformly chosen from $\{1,2,\ldots,m\}$, the probability that we made an error is at most $\frac{d}{m}$.

(b) The solution is $\left(\frac{d}{m}\right)^r$. We made an error if $F(x)$ is not identical to $G(x)$ and all $n_1,\ldots,n_r$ are roots of $F(x)-G(x)$. By part (a), the probability that n_i is the root of $F(x)-G(x)$ is at most $\frac{d}{m}$ for $i=1,\ldots,r$. Since $n_1,\ldots,n_r$ are independently chosen, the probability that all $n_1,\ldots,n_r$ are roots of $F(x)-G(x)$ is at most $\left(\frac{d}{m}\right)^r$.

4. (a) $X \sim \mathsf{Binomial}(n, \frac{1}{m})$; the limiting distribution of X is Pois(α) whose probability mass function is $p(k) = \frac{\alpha^k}{k!}\exp(-\alpha)$.

   (b) Since $X \sim \text{Bin}(n, \frac{1}{m})$, so $\mathbb{P}(X = 0) = (1 - \frac{1}{m})^n$. Define Bernoulli random variable Y_i such that $Y_i = 1$ if bin i is empty and $Y_i = 0$ if bin i is not empty. Then $\mathbb{P}[Y_i = 1] = \mathbb{P}[X = 0] = (1 - \frac{1}{m})^n$. Since the fraction of empty bins is given by $\frac{1}{m}\sum_{i=1}^m Y_i$, by the linearity of the expectation, we get the the fraction of empty bins on average is $\frac{1}{m}\sum_{i=1}^m \mathbb{E}[Y_i] = (1 - \frac{1}{m})^n$.

   In the limit as $m \to \infty$, $X \sim \text{Pois}(\alpha)$ and thus $\mathbb{P}(X = 0) = \exp(-\alpha)$. Then the fraction of empty bins on average is $\exp(-\alpha)$.

   (c) First consider all the possible assignments of balls into bins. For the i-th ball, it could land in any one of m bins and thus there are m possible assignments for it. In total, there are m^n possible assignments of n balls into m bins. Next consider all the possible assignments of balls into bins such that all bins contain exactly α balls. For the first bin, since it contains α balls, there are $\binom{n}{\alpha}$ different choices of balls into the first bin. Then consider the second bin, since it contains α balls and there are $n-\alpha$ balls left unassigned, there are $\binom{n-\alpha}{\alpha}$ different choices of balls into the second bin. Continue this argument for all the bins, and in total there are $\binom{n}{\alpha}\binom{n-\alpha}{\alpha}\cdots\binom{\alpha}{\alpha} = \frac{n!}{(\alpha!)^m}$ possible assignments of balls into bins such that all bins contain exactly α balls. In conclude, the probability that all bins contain exactly α balls is $\frac{n!}{(\alpha!)^m m^n}$.

   (d) The probability of seeing at least one ball from every bin after m rounds is $\frac{\alpha^m m!}{n^m} = \frac{m!}{m^m}$. For each round, there are n different choices of balls, so in m rounds there are n^m different choices of balls. To see at least one ball from every bin after m rounds, we need exactly 1 ball from every bin. Since each bin contains exactly α balls, there are α^m different choices of m balls. Furthermore, the order of seeing these m balls could be arbitrarily and there are $m!$ different orders of these m balls.

   Another way to derive is as follows. To see at least one ball from every bin after m rounds means that each round we pick a ball from a bin that is different from what we've picked, so the answer is given by $\frac{m}{m}\frac{m-1}{m}\frac{m-2}{m}\cdots\frac{1}{m} = \frac{m!}{m^m}$.

   (e) Let T_i denote the number of additional rounds to see at least one ball from i bins, given that we have seen at least one ball from $i-1$ bins. Then T_i is distributed as geometric distribution with parameter $\frac{m-(i-1)}{m}$. Therefore, $\mathbb{E}[\sum_{i=1}^m T_i] = \sum_{i=1}^m \frac{m}{m-(i-1)} = m\sum_{i=1}^m \frac{1}{i}$.

5. (a) Each needle is lying in the circle with probability $\frac{\pi}{4}$ and thus $X \sim \text{Bin}(n, \frac{\pi}{4})$. Hence, $\mathbb{P}[X = k] = \binom{n}{k}\left(\frac{\pi}{4}\right)^k\left(1 - \frac{\pi}{4}\right)^{n-k}$ for $1 \le k \le n$.

   (b) Let $\widehat{\pi}$ denote the maximum likelihood estimator of π. Then $\widehat{\pi} = \arg\max_x \binom{n}{k}\left(\frac{x}{4}\right)^k\left(1 - \frac{x}{4}\right)^{n-k}$. Hence, $\widehat{\pi} = \frac{4k}{n}$.

   (c) Note that the mean of X is $n\frac{\pi}{4}$. To estimate π within 0.1 means to estimate $\frac{\pi}{4}$ with 0.025. Using Chebychev inequality, we have

$$\mathbb{P}\left\{\frac{\pi}{4} \in \left[\frac{X}{n} - \frac{a}{2\sqrt{n}}, \frac{X}{n} + \frac{a}{2\sqrt{n}}\right]\right\} \ge 1 - \frac{1}{a^2}.$$

   Pick $a = 5$ such that $1 - \frac{1}{a^2} = 96\%$, and we need $\frac{a}{2\sqrt{n}} = 0.025$. Solving for n, we get $n = 10^4$.