University of Illinois Summer 2014

ECE 313: Exam II

Thursday, July 24, 2014
7:00 p.m. — 8:15 p.m.
269 Everitt Lab

1. (a) Note that $X \sim \text{Bin}(n, 1/2)$. Let $\tilde{X} \sim \mathcal{N}(n/2, n/4)$. It follows from the Gaussian approximation that

$$
\begin{aligned}
P\left\{|\frac{X}{n} - 0.5| \geq 0.1\right\} &= P\left\{|\frac{X - 0.5n}{\sqrt{n/4}}| \geq \frac{0.1n}{\sqrt{n/4}}\right\} \\
&\approx P\left\{|\frac{\tilde{X} - 0.5n}{\sqrt{n/4}}| \geq 0.2\sqrt{n}\right\} \\
&= 2Q\left(0.2\sqrt{n}\right) = 2 - 2\Phi\left(0.2\sqrt{n}\right).
\end{aligned}
$$

   (b) The Chebychev inequality implies that $P\left\{|\frac{X}{n} - 0.5| \geq 0.1\right\} \leq \frac{0.25n}{(0.1n)^2} = \frac{25}{n}$.

2. (a) Let $Y = -\ln X$, then Y takes values in $(-\infty, \infty)$. Fix $c \in \mathbb{R}$, then

$$F_Y(c) = \mathbb{P}[Y \leq c] = \mathbb{P}[-\ln X \leq c] = \mathbb{P}[X \geq e^{-c}] = \exp(-\exp(-c)).$$

   (b) Let U denote the uniform random variable supported over $[0, 1]$. We need to choose an appropriate function g such that $g(U)$ follows the Gumbel distribution, i.e., the CDF of $g(U)$ equals F. We propose to use $g(u) = F^{-1}(u) = -\ln(-\ln(u))$. Here is another way to find g. For part (a), we know $-\ln X$ has the required CDF if $X \sim \exp(1)$. Recall that we use $-\ln U$ to generate $\exp(1)$. Therefore, $-\ln(-\ln U)$ follows the Gumbel distribution.

3. (a) Since T_1 and T_2 are independently and exponentially distributed with the same unknown parameter λ, their joint pdf is given by $f_{T_1,T_2}(u, v) = \lambda^2 \exp(-\lambda(u + v))$ if $u, v > 0$ and $f_{T_1,T_2}(u, v) = 0$ otherwise.

   (b) By definition of the maximum likelihood estimation,

$$\hat{\lambda} = \arg\max_{\lambda} f_{T_1,T_2}(t_1, t_2) = \arg\max_{\lambda} \lambda^2 \exp(-\lambda(t_1 + t_2)) = \frac{2}{t_1 + t_2}.$$

   (c) By the definition of failure rate function,

$$h(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\mathbb{P}[t < T \leq t + \epsilon | T > t] \overset{(a)}{=} \lim_{\epsilon \to 0} \frac{1}{\epsilon}\mathbb{P}[0 < T \leq \epsilon] = f_T(0) = \lambda,$$

   where (a) follows from the memoryless property. Only partial credits will be given if just using the formula $h(t) = \frac{f_T(t)}{1 - F_T(t)}$.

4. (a) Define $\Lambda(u) = \frac{f_1(u)}{f_0(u)}$, where $f_1(u) = \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(x-1)^2}{8}\right)$ and $f_0(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x+1)^2}{2}\right)$. Therefore,

$$\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \frac{1}{2} \exp\left(-\frac{(x-1)^2}{8} + \frac{(x+1)^2}{2}\right) = \frac{1}{2} \exp\left(\frac{3}{8}x^2 + \frac{5}{4}x + \frac{3}{8}\right).$$

(b) The MAP rule says H_1 if $\Lambda(u) > \frac{P(H_0)}{P(H_1)} = \frac{1}{2}$ and says H_0 otherwise. Plugging the formula of $\Lambda(u)$ and simplifying the answer, we get that the MAP rule says H_1 if $3u^2+10u+3 \geq 0$, i.e., $u \geq -\frac{1}{3}$ or $u \leq -3$ and says H_0 otherwise.

(c) By definition of the probability of miss,

$$\begin{aligned} p_{\text{miss}} &= \mathbb{P}[\text{MAP says } H_0|H_1] = \mathbb{P}[a \leq X \leq b|H_1] = \mathbb{P}[X \leq b|H_1] - \mathbb{P}[X \leq a|H_1] \\ &= \mathbb{P}\left[\frac{X-1}{2} \leq \frac{b-1}{2}|H_1\right] - \mathbb{P}\left[\frac{X-1}{2} \leq \frac{a-1}{2}|H_1\right] = \Phi\left(\frac{b-1}{2}\right) - \Phi(\frac{a-1}{2}) \\ &= Q\left(\frac{1-b}{2}\right) - Q\left(\frac{1-a}{2}\right). \end{aligned}$$

1!

5. (a) The number of arrivals during time interval [0, 2] has the Poisson distribution with parameter 2λ, and hence there is exactly one customer arriving up to time $t = 2$ with probability $\frac{(2\lambda)\ \exp(-2\lambda)}{1!}$.

(b) Let T_2 denote the arrival time of the second customer arrival. Then T_2 follows the Erlang distribution with mean given by $\frac{2}{\lambda}$.

(c) We need to calculate the following conditional probability:

$$\begin{aligned} \mathbb{P}[N_1 = 3|N_3 = 5] &\overset{(a)}{=} \frac{\mathbb{P}[N_1 = 3, N_3 = 5]}{\mathbb{P}[N_3 = 5]} \overset{(b)}{=} \frac{\mathbb{P}[N_1 = 3, N_3 - N_1 = 2]}{\mathbb{P}[N_3 = 5]} \overset{(c)}{=} \frac{\mathbb{P}[N_1 = 3]\mathbb{P}[N_3 - N_1 = 2]}{\mathbb{P}[N_3 = 5]} \\ &\overset{(d)}{=} \frac{\lambda^3 \exp(-\lambda)(2\lambda)^2 \exp(-2\lambda)5!}{3!2!(3\lambda)^5 \exp(-3\lambda)} = \binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2, \end{aligned}$$

where (a) follows from the definition of the conditional probability; (b) holds due to $\{N_1 = 3, N_3 = 5\} = \{N_1 = 3, N_3 - N_1 = 2\}$; (c) holds because N_1 and $N_3 - N_1$ are independent; (d) holds because $N_t - N_s \sim \text{Pois}(\lambda(t-s))$.

6. (a) By definition, $f_X(u) = \int_{-\infty}^{+\infty} f_{X,Y}(u,v)dv$. Therefore,

$$f_X(u) = \begin{cases} \int_0^1 0.75dv = 0.75 & \text{if } 0 \leq u \leq 1 \\ \int_0^1 0.25dv = 0.25 & \text{if } 3 \leq u \leq 4, \\ 0 & \text{else .} \end{cases} \tag{1}$$

(b) By definition, $f_{X|Y}(u|0.5) = \frac{f_{X,Y}(u,v)}{f_Y(0.5)}$. Notice that $f_Y(0.5) = \int_0^1 0.75 + \int_3^4 0.25 = 1$ and thus

$$f_{X|Y}(u|0.5) = \frac{f_{X,Y}(u,0.5)}{f_Y(0.5)} = \begin{cases} 0.75 & \text{if } 0 \leq u \leq 1 \\ 0.25 & \text{if } 3 \leq u \leq 4, \\ 0 & \text{else .} \end{cases} \tag{2}$$

(c) Yes. Note that $f_Y(v) = 1$ if $0 \leq v \leq 1$ and $f_Y(v) = 0$ otherwise. Hence $f_{X,Y}(u,v) = f_X(u)f_Y(v)$ and thus X is independent of Y. We cannot conclude that X and Y are independent by just pointing out that the support of $f_{X,Y}$ is a product set, and only partial credits will be given if this is the case. Also, we cannot conclude that X and Y are independent by just pointing out that $f_{X|Y}(u|0.5) = f_X(u)$, and only partial credits will be given if this is the case.

(d) Let S denote the support of $f_{X,Y}$ and R denote the region $\{(u,v) : u \leq v\}$. Notice that

$$P[X \leq Y] = \mathbb{P}[(X,Y) \in R] = \int_R f_{X,Y}(u,v)dudv = \int_{R \cap S} f_{X,Y}(u,v)dudv$$
$$\overset{(a)}{=} \int_{(u,v)\in[0,1]^2 : u \leq v} 0.75dudv = 0.75 \times \text{area}\left(\{(u,v) \in [0,1]^2 : u \leq v\}\right) = \frac{3}{8},$$

where (a) follows because $R \cap S = \{(u,v) \in [0,1]^2 : u \leq v\}$ and $f_{X,Y}(u,v) = 0.75$ if $(u,v) \in R \cap S$.