University of Illinois Summer 2014

ECE 313: Exam II

Thursday, July 24, 2014
7:00 p.m. — 8:15 p.m.
269 Everitt Lab

1. **[10 points]** Suppose X has the binomial distribution $\mathrm{Bin}(n, \frac{1}{2})$ for some $n \geq 1$.

   (a) (6 points) Use the Gaussian approximation to find an approximation to $P\left\{|\frac{X}{n} - 0.5| \geq 0.1\right\}$. Use of the continuity correction is not required. If appropriate, use the Φ or Q function.

   (b) (4 points) Find the Chebychev upper bound on $P\left\{|\frac{X}{n} - 0.5| \geq 0.1\right\}$.

2. **[12 points]** Generating Gumbel Distribution.

   (a) (6 points) Suppose X has the exponential distribution with mean 1. Find the CDF of $-\ln X$.

   (b) (6 points) Give a procedure that transforms a random variable U that is uniformly distributed on the interval [0,1] into a random variable with CDF $F(c) = \exp(-\exp(-c))$ for $c \in \mathbb{R}$. (Such CDF is known as the Gumbel distribution, and your procedure could be used to generate Gumbel distributed random variable for a simulation.)

3. **[18 points]** A company produces a batch of light bulbs and would like to estimate the failure rate (assumed to be constant). Assume that the life time for all light bulbs is independently and exponentially distributed with the same unknown parameter λ.

   (a) (6 points) Let T_1, T_2 be the life times of two light bulbs. Find the joint pdf of T_1 and T_2.

   (b) (6 points) Suppose that the joint pdf of T_1 and T_2 is $f_{T_1,T_2}(u, v) = \lambda^a \exp(b\lambda(u + v))$ for some constants a, b with $a > 0, b < 0$. Derive the maximum likelihood estimator of λ, given observations $T_1 = t_1$ and $T_2 = t_2$. Show your derivation.

   (c) (6 points) Using the definition of failure rate function, derive the failure rate function of T_1.

4. **[18 points]** There are two hypotheses, H_1 and H_0. Suppose if H_0 is true, the observation X is distributed as $\mathcal{N}(-1, 1)$; if H_1 is true, the observation X is distributed as $\mathcal{N}(1, 4)$. Assume the prior probabilities $P(H_0) = \frac{1}{3}$ and $P(H_1) = \frac{2}{3}$.

   (a) (6 points) Find the likelihood ratio $\Lambda(u)$. Simplify your answer as much as possible.

   (b) (6 points) Express the MAP rule for deciding which hypothesis is true, given X, in as simple a way as possible.

   (c) (6 points) Suppose the MAP rule reduces to declaring H_0 is true if $a \leq X \leq b$ for some constants a, b with $a < b$. Express the probability of miss for the MAP rule in terms of a, b, and either the Φ or Q function.

5. **[18 points]** Suppose customers arrive at a restaurant starting at time zero according to a Poisson process $(N_t : t \geq 0)$ with rate λ, where N_t denotes the number of customers arriving up to time t.

   (a) (6 points) What is the probability there is exactly one customer arriving *up to* time $t = 2$?

   (b) (6 points) What is the mean arrival time of the second customer?

(c) (6 points) Conditioned on the event five customers have arrived during the time interval $[0, 3]$, find the probability three customers have arrived before $t = 1$.

6. **[24 points]** Let X and Y be random variables with a joint pdf given by

$$f_{X,Y}(u,v) = \begin{cases} 0.75 & \text{if } 0 \leq u \leq 1, 0 \leq v \leq 1 \\ 0.25 & \text{if } 3 \leq u \leq 4, 0 \leq v \leq 1, \\ 0 & \text{else .} \end{cases}$$

(a) (6 points) Find the marginal pdf $f_X(u)$. Be sure to specify it for $-\infty < u < +\infty$.

(b) (6 points) Find the conditional pdf $f_{X|Y}(u|0.5)$. Be sure to specify it for $-\infty < u < +\infty$.

(c) (6 points) Are X and Y are independent? Prove your answer.

(d) (6 points) Find the numerical value of $P[X \leq Y]$.