Problem 1

a) False

$$P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$$

b) True

Definition 2.4.6 of the textbook

c) False

$$E\left[X + Y^2\right] = E\left[X\right] + E\left[Y^2\right]$$

d) True

$$\sum_{i=1}^{\infty}(e-1)e^{-i} = (e-1)\sum_{i=1}^{\infty}e^{-i} = (e-1)\left(\frac{1}{e}+\frac{1}{e^2}+...\right) = (e-1)\left((1+\frac{1}{e}+\frac{1}{e^2}+...)-1\right) = (e-1)\left(\frac{1}{1-1/e}-1\right) =$$

$$= (e-1)\left(\frac{e}{e-1}-1\right) = (e-1)\left(\frac{1}{e-1}\right) = 1$$

e) False
The equality holds only if X and Y are independent (Problem 6 of the Problem Set 5)

f) False

$P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B) \leq P(A) + P(B)$, but $P(A) + P(B) > 1$. The maximum value of $P(A \bigcup B)$ (i.e., 1) occurs when $P(A \bigcap B)$ is equal to

(3/4+2/3)-1=5/12

Hence, the smallest possible value for $P(A \bigcap B)$ is equal to 5/12.

g) True

$$A \setminus (B \bigcap C) = A \bigcap (B \bigcap C^C)^C = A \bigcap (B^C \bigcup C) = (A \bigcap B^C) \bigcup (A \bigcap C) = (A \setminus B) \bigcup (A \setminus C^C)$$

h) False
The maximum is obtained at 1.

i) True

$$P(both\,children\,are\,boys \mid at\,least\,one\,is\,boy) = \frac{|\{BB\}|}{|\{BB, BG, GB\}|} = \frac{1}{3}$$

or put another way,

$$P(both\,children\,are\,boys \mid at\,least\,one\,is\,boy) = \frac{P(both\,children\,are\,boys)}{P(at\,least\,one\,is\,boy)} = \frac{1/4}{3/4} = \frac{1}{3}$$

j) False

Probability that A wins the series in 4 games=

$$\binom{3}{2}(2/3)^2(1/3)(2/3) = 3(2/3)^3(1/3) = (2/3)^3 = 8/27$$

Problem 2

a) The probability that “the ace of spades and the ace of hearts are in the same pile” is equal to 12/51: Ace of spades is in some pile. We should find the probability that ace of hearts is in the same pile. To have this, ace of hearts must one of the other 12 cards of that pile. On the other hand ace of hearts has 51 possible positions.
b) 12/25: This is similar to the above but this time ace of hearts can be any of the remaining 25 cards since it is not in two of the piles.

Problem 3

There are four cases to consider:

- All three numbers are equal to i. There is one such case.
- Two of the three numbers are i, and the other is less than i: we have 3 ways to choose which number is less than i and then we have i − 1 options for the value of this number. So there are 3 (i − 1) such cases.
- Two of the three numbers are i, and the other is more than i: we have 3 ways to choose which number is more than i and then we have 6-i options for the value of this number. So there are 3 (6-i) such cases.
- The three numbers are different. Consider one number is i. There are 6-i options for the value of the number more than i. There are i-1 options for the value of the number less than i. here are six combinations of these three different numbers as well. Therefore, for this case we will have: 6 (i-1) (6-i) ways

Hence, $|\{X = i\}| = 1 + 3(i-1) + 3(6-i) + 6(i-1)(6-i)$

$$P_X(i) = \frac{1 + 3(i-1) + 3(6-i) + 6(i-1)(6-i)}{6^3} = \frac{16 + 6(i-1)(6-i)}{6^3}$$ For i=1, 2, 3, 4, 5, 6

$P_X(1) =$ 0.074074074

$P_X(2) =$ 0.185185185

$P_X(3) =$ 0.240740741

$P_X(4) =$ 0.240740741

$P_X(5) =$ 0.185185185

$P_X(6) =$ 0.074074074

Problem 4

From the information given in the problem, we obtain the following Karnaugh map:

	College of Engineering	Non-College of Engineering
Male	8000	13200
Female	2000	16800

$|UIUC| = 40000$

$|College\,of\ Eng.| = 10000$

a) The probability that the student is male given that the student is not in the college of engineering:

$$= P(M \mid NEng) = \frac{|M, NEng|}{|NEng|} == \frac{13200}{13200 + 16800} = 0.44$$

b) The probability that the student is in the college of engineering if the student is female:

$$= P(Eng \mid F) = \frac{|Eng, F|}{|F|} = \frac{2000}{2000 + 16800} = 0.11$$

Problem 5

a) The probability that X = 7 given that the first four rolls are 3,2,5,6:

In the first four rolls two of the three even numbers appeared. Therefore, only number 4 remained, and we want to have this happen in the 7th role. Four rolls have already happened. Hence, We have 3 more rolls remained. Since we want to have number 4 in the last roll, rolls number 5 and 6 can be any number except for 4 (i.e., (1, 2, 3, 5, 6)). Therefore for the three remaining rolls we must have:

Roll # 5	Roll # 6	Roll # 7
Any of (1,2,3,5,6)	Any of (1,2,3,5,6)	4

The probability is equal to: (5/6)(5/6)(1/6)=25/216

b) $E[X]$

This is similar to the coupon collector problem discussed in class. Let Y denote the number of rolls it takes to observe the first even number. Let Z denote the additional number of rolls needed to observe the second even number and finally, let W denote the additional number of rolls needed to see the third even number. We have Y~Geo(3/6), Z~Geo(2/6), W~Geo(1/6) and X=Y+Z+W.

$$E[X] = E[Y] + E[Z] + E[W] = 6/3 + 6/2 + 6/1 = 11$$

c) Since Y, Z, and W are independent:

$$Var[X] = Var[Y] + Var[Z] + Var[W] = 2 + 6 + 30 = 38$$