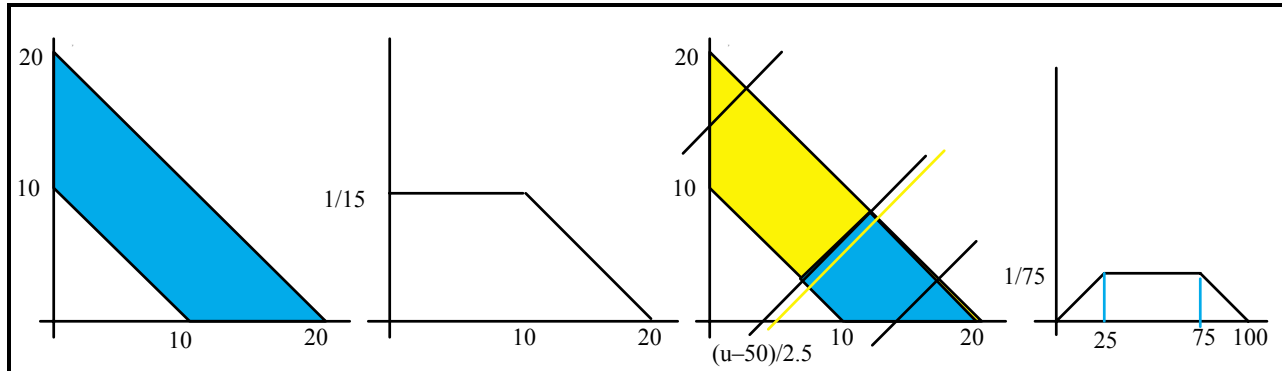


- 1.(a) The joint pdf has constant value K on the shaded region shown in the left-hand figure below. The area can be computed as the difference in area of two triangles as $(1/2) \cdot 20 \cdot 20 - (1/2) \cdot 10 \cdot 10 = 150$, giving $K = 1/150$.

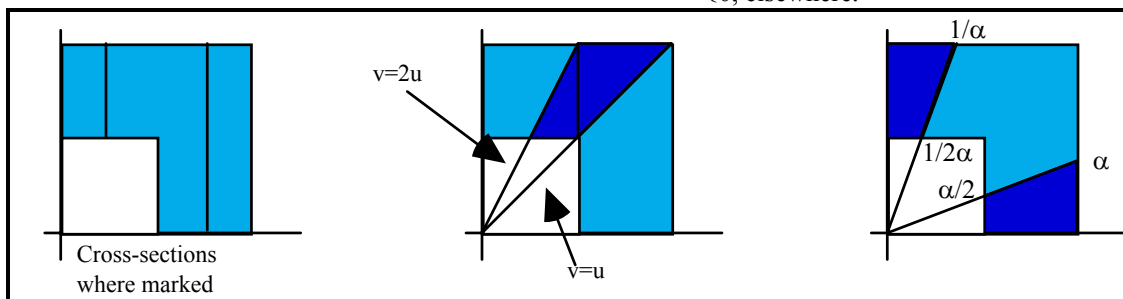


- (b) $f_R(x)$, the marginal pdf of R is the cross-sectional area of the joint pdf at x . Now, the cross-section is always a rectangle of height $1/150$, and its base is 10 if $0 < x < 10$, while the base is $20-x$ if $10 \leq x < 20$. Hence, we get that $f_R(x) = \begin{cases} 1/150, & 0 < x < 10, \\ (20-x)/150, & 10 \leq x < 20, \\ 0, & \text{elsewhere.} \end{cases}$ which is shown in the middle figure above. Note that $P\{\text{student spends less than 10 hours studying for the ECE 313 Final}\} = 2/3$.
- (c) T takes on values from 0 (when $S = 0$ and $R = 20$, that is, the student is sleepless and hopelessly confused) to 100 (when $S = 20$ and $R = 0$, that is, the student is well-rested and relies entirely on work done during the semester to carry him/her through the Final Exam). For any u , $0 < u < 100$, $P\{T \leq u\} = P\{S - R \leq (u-50)/2.5\} = (1/150) \cdot \text{shaded trapezoidal area in third figure above}$, but note that there are special cases to be considered when $u < 25$ and $u > 75$. We can now proceed to find $P\{T \leq u\} = F_T(u)$ and differentiate to find the pdf, but an easier argument is as follows. For $25 \leq u < 75$, and small δu , $P\{u \leq T \leq u + \delta u\} \approx f_T(u) \cdot \delta u$
 $= \text{volume between the two lines close together in the diagram} = (1/150) \cdot (10/\sqrt{2}) \cdot (\delta u/\sqrt{2})$, since that thin sliver has height $(1/150)$, and the sides of the base are $10/\sqrt{2}$ and $(\delta u/2.5)/\sqrt{2}$ respectively. We conclude that $f_T(u) = 1/75$ for $25 < u < 75$. A similar argument shows that the pdf of T increases linearly from 0 at $u = 0$ to $1/75$ at $u = 25$, and decreases from $1/75$ at $u = 75$ to 0 at $u = 100$. In short, it looks as shown above in the rightmost figure.

- 2.(a) The marginal pdf of X is the area of the cross-section of the joint pdf at the point u . There are two cases to be considered ($0 < u < 1/2$) and ($1/2 < u < 1$) as marked in the left-hand figure below. By inspection, we get that the area is $(4/3) \times (1/2) = 2/3$ for $0 < u < 1/2$, and $(4/3) \times 1 = 4/3$ for $1/2 < u < 1$, and 0 for all other u .

$$\text{Thus, } f_X(u) = \begin{cases} 2/3, & 0 < u < 1/2, \\ 4/3, & 1/2 < u < 1, \\ 0, & \text{elsewhere.} \end{cases} \quad \text{It is easily checked that the area under the pdf is 1.}$$

- (b) From the symmetry of the joint pdf, it is obvious that $f_Y(v) = \begin{cases} 2/3, & 0 < v < 1/2, \\ 4/3, & 1/2 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$



- (c) The probability desired is the volume in the darkly shaded region shown in the middle figure above. We can easily get the area of this region as the sum of two triangles $(1/2) \times (1/2) \times (1/4) + (1/2) \times (1/2) \times (1/2)$

= 3/16 and the probability is $(4/3) \times (3/16) = 1/4$. Anti-segregationists (i.e. those who believe in integration)

can get the same answer by writing $P\{X < Y < 2X\} = \int_{v=1/2}^1 \int_{u=v/2}^v 4/3 \, du \, dv = \int_{v=1/2}^1 2v/3 \, dv = v^2/3 \Big|_{1/2}^1 = \frac{1}{4}$.

- (d) The conditional pdf of X given $Y = \alpha$ is the cross-section $f_{X|Y}(u, \alpha)$ of the joint pdf, normalized to have unit area. In this instance, it is obvious that the cross-section is a rectangle, and hence given $Y = \alpha$, the conditional pdf of X is uniform on $(1/2, 1)$ if $0 < \alpha < 1/2$, and uniform on $(0, 1)$ if $1/2 < \alpha < 1$.

Thus, if $0 < \alpha < 1/2$, then

while if $1/2 < \alpha < 1$, then

$$f_{X|Y}(u|\alpha) = \begin{cases} 2, & 1/2 < u < 1, \\ 0 & \text{otherwise,} \end{cases} \quad f_{X|Y}(u|\alpha) = \begin{cases} 1, & 0 < u < 1, \\ 0 & \text{otherwise,} \end{cases}$$

The theorem of total probability gives $f_X(u) = \int_0^1 f_{X|Y}(u|\alpha) f_Y(\alpha) d\alpha$

$$= \int_0^{1/2} f_{X|Y}(u|\alpha) (2/3) d\alpha + \int_{1/2}^1 f_{X|Y}(u|\alpha) (4/3) d\alpha \quad \text{on substituting the numerical value of } f_Y(\alpha) d\alpha \text{ from part (b).}$$

Now, for fixed u , $0 < u < 1/2$, $f_{X|Y}(u|\alpha) = 0$ if $0 < \alpha < 1/2$, and $f_{X|Y}(u|\alpha) = 1$ if $1/2 < \alpha < 1$. Hence, the two integrals are 0 and $(1/2) \cdot (4/3)$, giving that $f_X(u) = 2/3$ for all numbers u , $0 < u < 1/2$. On the other hand, for fixed u , $1/2 < u < 1$, $f_{X|Y}(u|\alpha) = 2$ if $0 < \alpha < 1/2$, and $f_{X|Y}(u|\alpha) = 1$ if $1/2 < \alpha < 1$. Hence, the two integrals are $(1/2) \cdot 2 \cdot (2/3)$ and $(1/2) \cdot (4/3)$, giving that $f_X(u) = 4/3$ for all numbers u , $1/2 < u < 1$. This is the same answer as in part (b).

$$3. \quad E[|X-Y|] = \int_0^1 \int_0^1 |u-v| \, du \, dv = 2 \int_{v=0}^1 \int_{u=0}^v (v-u) \, du \, dv = 1/3.$$

Most simply, $\text{var}(X-Y) = \text{var}(X) + \text{var}(Y) - 2 \cdot \text{cov}(X, Y) = \text{var}(X) + \text{var}(Y)$ since X and Y are independent (and hence uncorrelated). Ross pp. 201-202 shows that $\text{var}(X) = \text{var}(Y) = 1/12$ and hence $\text{var}(X-Y) = 1/6$.

Otherwise, we note that $E[X-Y] = E[X] - E[Y] = 1/2 - 1/2 = 0$ since X and Y are uniformly distributed on

$$(0, 1). \quad \text{Hence, } \text{var}(X-Y) = E[(X-Y)^2] \text{ where } E[(X-Y)^2] = \int_0^1 \int_0^1 (u-v)^2 \, du \, dv = 1/3 \int_0^1 (1-v)^3 + v^3 \, dv = 1/6.$$

$$4.(a) \quad E[Z] = 2\{E[X^2] - E[Y^2]\} = 2\{\text{var}(X) - \text{var}(Y) + E^2[X] - E^2[Y]\} = -40.$$

$$(b) \quad \text{cov}(T, U) = \text{cov}(2X+Y, 2X-Y) = 4 \cdot \text{var}(X) - \text{var}(Y) = 7.$$

$$(c) \quad E[W] = E[3X+Y+2] = 3E[X] + E[Y] + 2 = 9.$$

$$\text{var}(W) = \text{var}(3X+Y+2) = \text{var}(3X+Y) \text{ (why?) } = 9 \cdot \text{var}(X) + \text{var}(Y) + 6 \cdot \text{cov}(X, Y) = 9 \cdot 4 + 9 + 6 \cdot (0.1 \cdot 2 \cdot 3) = 48.6.$$

$$(d) \quad P\{W > 0\} = 1 - \Phi((0-9)/\sqrt{48.6}) = 1 - \Phi(-9/\sqrt{48.6}) = \Phi(9/\sqrt{48.6}) \text{ (why?)}$$

$$5.(a) \quad \text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2 \cdot \text{cov}(X, Y) = 36$$

$$\text{var}(X-Y) = \text{var}(X) + \text{var}(Y) - 2 \cdot \text{cov}(X, Y) = 64. \quad \text{Hence, } \text{cov}(X, Y) = -7.$$

If $\text{var}(X) = 3 \cdot \text{var}(Y)$, then $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2 \cdot \text{cov}(X, Y) = 4 \cdot \text{var}(Y) - 14 = 36$, so that

$$\text{var}(Y) = 12.5, \text{var}(X) = 37.5, \text{ and } \rho = \text{cov}(X, Y) / \sqrt{\text{var}(X) \text{var}(Y)} = -7 / (12.5 \sqrt{3}).$$

- (b) If $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2 \cdot \text{cov}(X, Y) = \text{var}(X-Y) = \text{var}(X) + \text{var}(Y) - 2 \cdot \text{cov}(X, Y)$, then it must be that $\text{cov}(X, Y) = 0$ and hence X and Y are uncorrelated.

- (c) The values of $\text{var}(X)$ and $\text{var}(Y)$ do not affect the covariance computation at all. So, the random variables may or may not be uncorrelated.

6. It is easy to see that $f_X(u)$ has constant value $2/3$ for $0 < u < 1/2$, and constant value $4/3$ for $1/2 < u < 1$.

$$(a) \quad \text{Hence, } E[X] = \int_0^{1/2} u \cdot 2/3 \, du + \int_{1/2}^1 u \cdot 4/3 \, du = \frac{(1/2)^2}{3} - \frac{0^2}{3} + \frac{2 \cdot 1^2}{3} - \frac{2 \cdot (1/2)^2}{3} = \frac{7}{12}$$

$$\text{and } E[X^2] = \int_0^{1/2} u^2 \cdot 2/3 \, du + \int_{1/2}^1 u^2 \cdot 4/3 \, du = \frac{2 \cdot (1/2)^3}{9} - \frac{2 \cdot 0^3}{9} + \frac{4 \cdot 1^3}{9} - \frac{4 \cdot (1/2)^3}{9} = \frac{5}{12}.$$

$$\text{Therefore, } \text{var}(X) = E[X^2] - (E[X])^2 = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}.$$

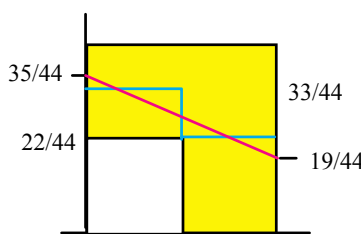
- (b) Since the pdf of Y is the same as the pdf of X , it has the same mean and variance.

$$(c) \quad E[XY] = \int_{u=0}^1 \int_{v=0}^1 uv \cdot 4/3 \, dv \, du - \int_{u=0}^{1/2} \int_{v=0}^{1/2} uv \cdot 4/3 \, dv \, du = \frac{4}{3} \left[\frac{1}{2} \cdot \frac{1}{2} - \frac{1}{8} \cdot \frac{1}{8} \right] = \frac{5}{16} \quad (\text{whoa! why?})$$

Hence, $\text{cov}(X, Y) = E[XY] - E[X]E[Y] = \frac{5}{16} - \frac{49}{144} = \frac{-4}{144} = \frac{-1}{36}$ and thus $\rho = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} = \frac{-4}{11}$.

$$(d) \quad b = \rho \sqrt{\text{var}(Y)/\text{var}(X)} = \rho = \frac{-4}{11}. \quad a = E[Y] - bE[X] = \frac{7}{12} \left[1 + \frac{4}{11} \right] = \frac{35}{44}. \quad \text{Hence,}$$

$\tilde{Y} = \frac{35}{44} - \frac{4X}{11} = \frac{35 - 16X}{44}$ which is a straight line from $(0, 35/44)$ to $(1, 19/44)$. The graphs of the best estimate (a step function) and the best linear (straight line) estimate are as shown. Notice that the straight line estimate passes through the mean point $(7/12, 7/12) = (E[X], E[Y])$ while the step function estimate does not. This is a characteristic of best linear estimators: if X equals its mean, the best linear estimate of Y is the mean of Y . Not always so for nonlinear optimum estimates. Notice also that the straight line estimate is a better approximation to the step function estimate in the interval $(1/2, 1)$ because that is where most of the probability mass is!



The two estimates $\hat{Y}$ and $\tilde{Y}$ are the same at $X = 1/8$ and $X = 13/16$.

$$(e) \quad \text{Since } \hat{Y} \text{ is a function of } X \text{ (not } Y\text{!) having value } 3/4 \text{ if } X < 1/2 \text{ and value } 1/2 \text{ if } X > 1/2, \text{ LOTUS allows us to write } E[(Y - \hat{Y})] = E[Y] - E[\hat{Y}] = \frac{7}{12} - \int_0^{1/2} (3/4)(2/3) \, du - \int_{1/2}^1 (1/2)(4/3) \, du = \frac{7}{12} - \frac{1}{4} - \frac{1}{3} = 0 \text{ so that}$$

the average error of the best mean-square estimate is 0. That $E[Y] = E[\hat{Y}]$ should be expected from Theorem 4.1 on p. 338 of the book. Similarly,

$$E[(Y - \tilde{Y})] = E[Y] - E[\tilde{Y}] = E[Y] - E\left[\frac{35 - 16X}{44}\right] = E[Y] - \frac{35}{44} - \frac{16}{44} E[X] = \frac{7}{12} - \frac{35}{44} - \frac{16}{44} \cdot \frac{7}{12} = 0.$$

$$\text{On the other hand, } E[(Y - \hat{Y})^2] = \int_{v=1/2}^1 \int_{u=0}^{1/2} (v - 3/4)^2 \cdot 4/3 \, dv \, du + \int_{v=0}^{1/2} \int_{u=1/2}^1 (v - 1/2)^2 \cdot 4/3 \, dv \, du$$

$$= (2/3) \int_{v=1/2}^1 (v - 3/4)^2 \, dv + (2/3) \int_{v=0}^{1/2} (v - 1/2)^2 \, dv = \frac{1}{16} = (0.25)^2 \text{ while}$$

$$E[(Y - \tilde{Y})^2] = E[(Y - a - bX)^2] = \text{var}(Y) \cdot [1 - \rho^2] = \left(\frac{11}{144}\right) \left[1 - \left(\frac{-4}{11}\right)^2\right] = \frac{35}{528} \approx (0.257\dots)^2 > \frac{1}{16} = (0.25)^2.$$

Ignoring X and estimating Y as $E[Y] = \frac{7}{12}$ leads to mean-square error $\text{var}(Y) = \frac{11}{144} \approx (0.276\dots)^2$