

1.	ω in	Prob.	Value of $X(\omega)$	
	$A^c B^c C^c$	1/8	0	(a) Hence, X takes on values $-1, 0, 1, 2$, and 3 .
	$A^c B^c C$	1/8	-1	
	$A^c B C^c$	1/8	2	(b) The pmf of X can be seen to be given by $p_X(-1) = 1/8$,
	$A^c B C$	1/8	1	$p_X(0) = 1/4 = p_X(1) = p_X(2)$, and $p_X(3) = 1/8$.
	$A B^c C^c$	1/8	1	The CDF of X can be readily written down from this data.
	$A B^c C$	1/8	0	
	$A B C^c$	1/8	3	
	$A B C$	1/8	2	

$$F(u) = \begin{cases} 0, & u < -1, \\ 1/8, & -1 \leq u < 0, \\ 3/8, & 0 \leq u < 1, \\ 5/8, & 1 \leq u < 2, \\ 7/8, & 2 \leq u < 3, \\ 1, & u \geq 3. \end{cases} \quad \text{Note that the function is right continuous!}$$

2.(a),(b) As studied in class, X is an exponential random variable with parameter μ . Hence, $P\{X > \tau\} = \exp(-\mu\tau)$ and $f_X(u) = \mu \cdot \exp(-\mu u)$ for $u > 0$, and 0 otherwise.

(c) The number of arrivals in the time interval $(0, 6]$ is a Poisson random variable $N(0,6]$ with parameter 6μ , and we get $P\{N(0,6] = 4\} = \frac{(6\mu)^4}{4!} \cdot \exp(-6\mu)$.

(d) The number of arrivals in $(4, 10]$ is $N(4,10]$ is also a Poisson random variable with parameter 6μ , and hence $P\{N(4,10] = 2\} = \frac{(6\mu)^2}{2!} \cdot \exp(-6\mu)$. We are now asked for $P\{N(0,6] = 4\} \cap \{N(4,10] = 2\}$. Now, $(0, 6]$ and $(4, 10]$ are intersecting intervals of time, but we can partition them into the disjoint intervals $(0, 4]$, $(4, 6]$, and $(6, 10]$ and thus write

$$\begin{aligned} P\{N(0,6] = 4\} \cap \{N(4,10] = 2\} &= P\{N(0,4] = 4\} \cap \{N(4,6] = 0\} \cap \{N(6,10] = 2\} \\ &+ P\{N(0,4] = 3\} \cap \{N(4,6] = 1\} \cap \{N(6,10] = 1\} \\ &+ P\{N(0,4] = 2\} \cap \{N(4,6] = 2\} \cap \{N(6,10] = 0\} \\ &= \left[\frac{(4\mu)^4}{4!} \cdot \exp(-4\mu) \right] \cdot \left[\exp(-2\mu) \right] \cdot \left[\frac{(4\mu)^2}{2!} \cdot \exp(-4\mu) \right] \\ &+ \left[\frac{(4\mu)^3}{3!} \cdot \exp(-4\mu) \right] \cdot \left[\frac{2\mu}{1!} \exp(-2\mu) \right] \cdot \left[\frac{4\mu}{1!} \cdot \exp(-4\mu) \right] \\ &+ \left[\frac{(4\mu)^2}{2!} \cdot \exp(-4\mu) \right] \cdot \left[\frac{(2\mu)^2}{2!} \cdot \exp(-2\mu) \right] \cdot \left[\exp(-4\mu) \right] \end{aligned}$$

because occurrences in disjoint intervals are independent and hence we can multiply the probabilities.

$$\text{Thus, } P\{N(0,6] = 4\} \cap \{N(4,10] = 2\} = \left(\frac{256}{3}\right) \cdot \exp(-10\mu) \cdot \left[\mu^6 + \mu^5 + \left(\frac{3}{16}\right)\mu^5\right].$$

(c) $P\{X > \tau \mid A\}$ is obviously 0 if $\tau \geq 6$, while for $0 \leq \tau < 6$,

$$P\{X > \tau \mid A\} = P\{\text{No arrivals in } (0, \tau] \mid 4 \text{ arrivals in } (0, 6]\}$$

$$= \frac{P\{0 \text{ in } (0, \tau] \cap 4 \text{ in } (0, 6]\}}{P\{4 \text{ in } (0, 6]\}} = \frac{P\{0 \text{ in } (0, \tau]\} P\{4 \text{ in } (\tau, 6]\}}{P(A)}$$

$$= \frac{[\exp(-\mu\tau) \cdot \exp(-\mu(6-\tau)) \cdot (\mu(6-\tau))^4 / 4!]}{\exp(-6\mu) \cdot (6\mu)^4 / 4!} = \frac{(6-\tau)^4}{6^4} = \left(\frac{6-\tau}{6}\right)^4 \text{ which has value 1 at } \tau = 0 \text{ and approaches 0}$$

as τ approaches 6.

3. X is $\mathcal{N}(500, 100^2)$. Thus,

(a) $P\{X < 700\} = \Phi((700-500)/100) = \Phi(2) = 0.9772$, so 97th percentile

(b) $\Phi(1.645) \approx 0.95$ so a "score" of 564.5 puts you at the 95th percentile.

(c) $P\{300 < X < 550\} = \Phi(550-500)/100 - \Phi((300-500)/100) = \Phi(0.5) - \Phi(-2) = \Phi(0.5) - (1 - \Phi(2))$
 $= \Phi(0.5) + \Phi(2) - 1 = 0.6915 + 0.9772 - 1 = 0.6687$

- 4.(a) $d/du(\exp(-u^2/2)) = -u \exp(-u^2/2)$. Hence,

$$E[|X|] = \int_{-\infty}^{\infty} |u| \cdot \phi(u) du = 2 \int_0^{\infty} u \cdot \phi(u) du = 2 \int_0^{\infty} u \cdot \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) du = \sqrt{\frac{2}{\pi}} \cdot \exp(-u^2/2) \Big|_0^{\infty} = \sqrt{\frac{2}{\pi}} < 1.$$

More generally, $E[|X-\mu|] = \sigma \sqrt{\frac{2}{\pi}}$ if the pdf of $\mathbf{X}$ is $\mathcal{N}(\mu, \sigma^2)$. In statistical applications, $E[|X-\mu|]$ is sometimes called the absolute error.

- (b) Since $t+x > t-x > 0$, $(t+x)(t-x) = t^2 - x^2 > (t-x)^2$. Hence, $\exp(-(t^2 - x^2)) < \exp(-(t-x)^2)$ and therefore

$$\exp\left(\frac{x^2}{2}\right) Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2 - x^2}{2}\right) dt < \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(t-x)^2}{2}\right) dt = 1/2. \text{ What? Why } 1/2?$$

- 5.(a) As in Problem 4(a), $E[|X|] = \sqrt{\frac{2}{\pi}}$

- (b) Since $P\{X > 0\} = 1/2$, the conditional pdf is just the right half of the Gaussian pdf with a doubling in height (to ensure that the area under the curve is 1), i.e. $f_{X|X>0}(u|X>0) = \begin{cases} 2 \cdot \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) & u > 0 \\ 0 & u \leq 0 \end{cases}$

- (c) $E[Y] = \int_0^{\infty} u \cdot \frac{2}{\sqrt{2\pi}} \exp(-u^2/2) du = \sqrt{\frac{2}{\pi}} \cdot \exp(-u^2/2) \Big|_0^{\infty} = \sqrt{\frac{2}{\pi}}$

- 6.(a) Obviously $P\{Y = \alpha\} = P\{Y = -\alpha\} = 1/2$.

- (b) $(1.29-1) = 0.29$. $(1.29-1)^2 = 0.0841$. $(\pi/4-1) = -0.214\dots$, $(\pi/4-1)^2 = 0.046\dots$.
 $(-\pi/4-(-1)) = -0.214\dots$, $(-\pi/4-(-1))^2 = 0.046\dots$. Note that the error for $+\mathbf{X}$ is the same as that for $-\mathbf{X}$.

- (c) $E[Z] = \int_0^{\infty} (u-\alpha)^2 f(u) du + \int_{-\infty}^0 (u+\alpha)^2 f(u) du = \int_{-\infty}^{\infty} (u^2 + \alpha^2) f(u) du - 4\alpha \int_0^{\infty} u f(u) du = 1 + \alpha^2 - 4\alpha/\sqrt{2\pi}$ on

expanding out the quadratics, changing variables, and using the fact that $E[X^2] = \sigma^2 + \mu^2 = 1$. Note that $uf(u)$ is a perfect integral. It is easy to show that $E[Z]$ has minimum value $1-2/\pi$ at $\alpha = \sqrt{2/\pi}$

- (d) From tables of $\Phi(\cdot)$, we get $P\{W = -3\} = P\{W = +3\} = \Phi(-2.5) = 0.0062$,
 $P\{W = 0\} = \Phi(0.5) - \Phi(-0.5) = 0.3830$, $P\{W = -1\} = P\{W = +1\} = \Phi(1.5) - \Phi(0.5) = 0.2417$, and
 $P\{W = -2\} = P\{W = +2\} = \Phi(2.5) - \Phi(1.5) = 0.0606$.

- (e) $P\{Z_2 = 1\} = P\{W < 0\} = 0.3085$.
 $P\{Z_1 = 1\} = P\{W = 2\} + P\{W = 3\} + P\{W = -1\} + P\{W = -2\} = 0.3691$
 $P\{Z_0 = 0\} = P\{W = 2\} + P\{W = 0\} + P\{W = -2\} = 0.5042 \neq P\{Z_0 = 1\}$.

- 7.(a) The lifetime is an exponential random variable with parameter λ . Hence, the *average* lifetime is $\lambda^{-1} = (-\ln 0.999)^{-1} \approx 999.5$ weeks which is more than 19 years! The *median* lifetime is T where

- $P\{X > T\} = \exp(-\lambda T) = 1/2$, i.e. $T = \lambda^{-1} \ln 2 \approx 692.8$ weeks which is only 13 years and a few months.

- (b) $P\{X > 1\} = \exp(-\lambda) = 0.999$ so even with this long-lived module, we can only be 99.9% sure that the thing is going to last for a week!

- (c) We can be sure that at least two of the three events $\{X_1 > t\}$, $\{X_2 > t\}$, $\{X_3 > t\}$ must have occurred. In fact, $\{Y > t\}$ is the (disjoint) union of the four events

$$\{X_1 > t\} \cap \{X_2 > t\} \cap \{X_3 > t\}, \quad \{X_1 > t\} \cap \{X_2 > t\} \cap \{X_3 \leq t\},$$

$$\{X_1 > t\} \cap \{X_2 \leq t\} \cap \{X_3 > t\}, \quad \text{and} \quad \{X_1 \leq t\} \cap \{X_2 > t\} \cap \{X_3 > t\}.$$

- (d) Since $P\{X_i > t\} = \exp(-\lambda t)$, we readily find that the probability of the first event in (c) is $\exp(-3\lambda t)$, while the other three have probability $\exp(-2\lambda t) [1 - \exp(-\lambda t)]$. Hence, $P\{Y > t\} = 3 \exp(-2\lambda t) - 2 \exp(-3\lambda t)$.

$$\text{Furthermore, } E[Y] = \int_0^{\infty} P\{Y > t\} dt = \int_0^{\infty} 3 \exp(-2\lambda t) - 2 \exp(-3\lambda t) dt = (3/2\lambda) - (2/3\lambda) = (5/6\lambda) < 1/\lambda$$

so that the **average** lifetime of the TMR system is actually smaller than that of the individual modules! To find the median lifetime, we set $P\{Y > T\} = 3 \exp(-2\lambda T) - 2 \exp(-3\lambda T) = 3p^2 - 2p^3 = 1/2$. It is easy to

show that $p = 1/2$ is one solution while the other two solutions correspond to $p < 0$ and $p > 1$. Hence, the median lifetime of the TMR system is also $\lambda^{-1} \ln 2 \approx 692.8$ weeks. Thus, in terms of these averages, the TMR system is just a waste of money; you have to pay more than three times as much (the majority-logic gate, not being one of the best things in life, is not free!) and the performance seems to be worse than that of a single chip.

- (e) $P\{Y > 1\} = 3 \exp(-2\lambda) - 2 \exp(-3\lambda) = 3(0.999)^2 - 2(0.999)^3 = 0.999997002$ which is *much better* than the comparable figure 0.999 for the single chip.
- (f) $P\{Y > t\} = 0.999$ for $t = -\lambda^{-1} \ln 0.98163 \approx 18.53$ weeks which is more than 4 months! Thus, we can be 99.9% sure that the TMR system will work for at least 18.53 weeks. This is **much** longer than the week that a single module might be expected to last with 99.9% probability. It is in **these** matters that the TMR system makes a difference, and why it is one of the choices available to the system designer who wants to create reliable systems. As noted above, the average lifetime of the TMR system is worse than that of the single chip. This is an illustration of the truism that complex systems have worse reliability than simple systems. What the TMR system is giving up is average lifetime in return for **much** higher reliability in early life.

- 8.(a),(b) $f_Y(t) = -\frac{d}{dt} P\{Y > t\} = 6\lambda[\exp(-2\lambda t) - \exp(-3\lambda t)]$ which has value 0 at $t = 0$ but is increasing (for small values of t) while $f_X(t) = \lambda \exp(-\lambda t)$ has value λ at $t = 0$ but decays away forever thereafter. Thus, for small values of t , $f_Y(t) < f_X(t)$. The two functions are equal if $6[\exp(-2\lambda t) - \exp(-3\lambda t)] = \exp(-\lambda t)$, i.e. at the roots $(6 \pm \sqrt{12})/12$ of the quadratic $6x^2 - 6x + 1$ where $x = \exp(-\lambda t)$. We get that $f_Y(t) < f_X(t)$ for $t < 237.28 \dots$ weeks or $t > 1553.58 \dots$ weeks. In between these values, $f_Y(t) > f_X(t)$.

- (c) $h_Y(t) = f_Y(t)/P\{Y > t\} = 6\lambda \left[\frac{1 - \exp(-\lambda t)}{3 - 2\exp(-\lambda t)} \right]$ has value 0 at $t = 0$ and is a monotone increasing function that approaches 2λ as $t \rightarrow \infty$. In contrast, $h_X(t) = \lambda$ for all t . The two functions cross at $t = 287.54$ weeks = 5.53 years.

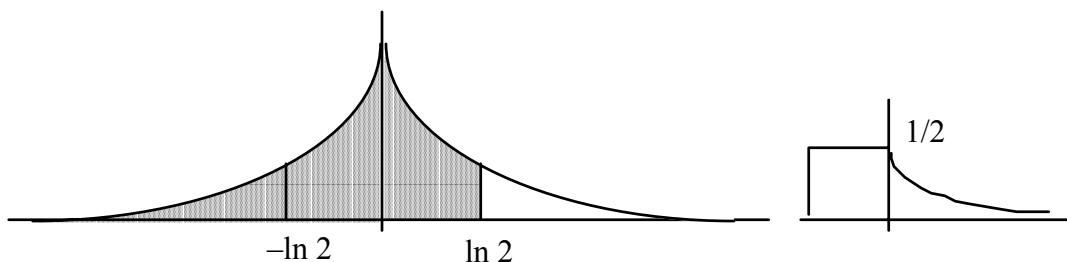
- (d),(e) $h_Y(1) < h_X(1)$ and therefore, the one-week old TMR system is less likely to fail in the next minute. On the other hand, $h_Y(1040) > h_X(1040)$ and hence the twenty-year old VLSI chip is less likely to fail. Similar observations in human beings have led to aphorisms such as "Never trust anyone over thirty."

- 9.(a) $E[X] = \int_0^\infty u f(u) du = \int_0^\infty u (bu) \exp(-bu^2/2) du = \sqrt{2/b} \int_0^\infty \sqrt{v} \exp(-v) dv$ on setting $bu^2/2 = v$. The integral is $\Gamma(3/2) = (1/2)\Gamma(1/2) = \sqrt{\pi}/4$ and hence $E[X] = \sqrt{2/b} \sqrt{\pi}/4 = \sqrt{\pi/2} \sqrt{1/b}$.

- (b) $E[X] = \int_0^\infty P\{X > t\} dt = \int_0^\infty \exp(-bt^2/2) dt$. But, we know that $\int_0^\infty (\sqrt{b/2\pi}) \exp(-bt^2/2) dt = 1/2$ and hence $E[X] = \sqrt{\pi/2} \sqrt{1/b}$ just as in part (a).

- (c) The median lifetime is T satisfying $P\{X > T\} = \exp(-bT^2/2) = 1/2$. This gives $T = \sqrt{2 \ln 2} \sqrt{1/b}$. Since $2 \ln 2 < \pi/2$, the median is smaller than the mean. The mode of the pdf is easily found to be at $\sqrt{1/b}$ and is the smallest of the three central measures.

- (d) $\frac{d}{db}(bt) \exp(-bt^2/2) = t \exp(-bt^2/2) - bt \exp(-bt^2/2) \cdot t^2/2$ is zero for $b = \sqrt{2}/t$. Thus, if we observe that $X = t$, the maximum-likelihood estimate of b is $\sqrt{2}/t$. Reality check: If t is large, we estimate the value of b to be quite small. This makes sense. If the system lasted for a long time, its hazard rate must be small, and the hazard rate is proportional to b .



10.(a) From the figure shown above, we see that the pdf is symmetric about $u = 0$. Hence, we get that

$$P\{X \leq \ln 2\} = \frac{1}{2} + \int_0^{\ln 2} \frac{1}{2} \exp(-u) du = \frac{1}{2} + \left[-\frac{1}{2} \exp(-u) \right]_0^{\ln 2} = \frac{3}{4}. \text{ Notice that } P\{0 \leq X \leq \ln 2\} = \frac{1}{4}$$

$$(b) \quad P\{|X| \leq \ln 2 \mid X \leq \ln 2\} = \frac{P\{|X| \leq \ln 2\} \cap \{X \leq \ln 2\}}{P\{X \leq \ln 2\}} = \frac{P\{|X| \leq \ln 2\}}{P\{X \leq \ln 2\}} = \frac{2P\{0 \leq X \leq \ln 2\}}{3/4} = \frac{1/2}{3/4} = \frac{2}{3}.$$

$$(c) \quad \cos(\pi X/2) = 0 \text{ if } X \text{ is an odd integer, and} \\ P\{\cos(\pi X/2) < 0\} = \dots + P\{-7 < X < -5\} + P\{-3 < X < -1\} + P\{1 < X < 3\} + P\{5 < X < 7\} + \dots \\ = 2[P\{1 < X < 3\} + P\{5 < X < 7\} + \dots] \text{ (by symmetry)} = [\exp(-1) - \exp(-3)] + [\exp(-5) - \exp(-7)] + \dots \\ = \exp(-1) \cdot \frac{1}{1 + \exp(-2)} = \frac{1}{2 \cosh(1)} = 0.324\dots \text{ on summing the geometric series.}$$

(d) The minimum value of I is -1 . $F_I(b) = P\{I \leq b\} = 0$ if $b < -1$.

For $b \geq -1$, $F_I(b) = P\{I \leq b\} = P\{e^V - 1 \leq b\} = P\{V \leq \ln(b+1)\} = F_V(\ln(b+1))$. But,

$$F_V(a) = \begin{cases} e^{a/2}, a \leq 0, \\ 1 - e^{-a/2}, a > 0. \end{cases} \quad \text{Thus, } F_I(b) = \begin{cases} \frac{b+1}{2} - 1 \leq b \leq 0, \\ 1 - \frac{1}{2(b+1)}, b > 0, \end{cases}$$

$$\text{and } f_I(b) = \begin{cases} \frac{1}{2} - 1 \leq b \leq 0, \\ \frac{1}{2(b+1)^2}, b > 0. \end{cases} \quad \text{Note that the pdf has constant value } 1/2 \text{ in the range } -1 \leq b \leq 0.$$