

1. (a) $F_X(1) = 1$, $F_X(3/2) = 3/4$. Thus, $F_X(u)$ is not a nondecreasing function and thus cannot be a valid CDF.

(b) Yes. $F_X(u)$ is a valid CDF, and is continuous except at $u = 0$.

$$\begin{aligned} P\{|X| > 0.5\} &= 1 - P\{|X| \leq 0.5\} = 1 - (F_X(0.5) - F_X(-0.5^-)) = 1 - (F_X(0.5) - F_X(-0.5)) \\ &= 1 - (1 - (1/4)\exp(-3/2) - (1/2)\exp(-1)) = (1/4)\exp(-3/2) + (1/2)\exp(-1). \end{aligned}$$

(c) $F_X(u)$ is not right continuous at $u = 0$ and thus cannot be a valid CDF.

2.(a) $P(\text{works for exactly 2 hours}) = P\{X = 2\} = F_X(2^+) - F_X(2^-) = 1/4$.

(b) $P(\text{works for more than 2 hours}) = P\{X > 2\} = 1 - F_X(2^+) = 1/4$.

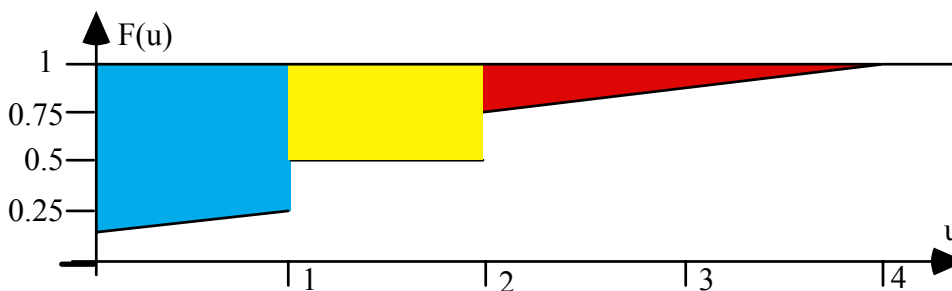
(c) $P(\text{works for less than 2 hours}) = P\{X < 2\} = F_X(2^-) = 1/2$.

(d) $P(\text{works for exactly 3 hours}) = P\{X = 3\} = F_X(3^+) - F_X(3^-) = 0$.

(e) $P(\text{works for more than 1/2 but less than 3 hours}) = P\{1/2 < X < 3\} = F_X(3^-) - F_X((1/2)^+) = 7/8 - 3/16 = 11/16$.

(f) $P(\text{works for more than 2 hours given that the student works at all}) = P\{X > 2 | X > 0\}$

$$= \frac{P\{X > 2\} \cap \{X > 0\}}{P\{X > 0\}} = \frac{P\{X > 2\}}{P\{X > 0\}} = \frac{1 - F_X(2^+)}{1 - F_X(0^+)} = \frac{1}{4} \times \frac{8}{7} = 2/7.$$



(g) The CDF is as shown, and as discussed in class, $E[X] =$ shaded area between CDF and the line at height 1. By elementary geometry, the shaded area is $1 \times (7/8 + 6/8)/2 + 1/2 + (1/2) \times (2) \times (1/4) = 22/16 = 1.375$ hours.

3.(a) $E[X] = \int_0^{\infty} P\{X > u\} du = \int_0^1 P\{X > u\} du + \int_1^2 P\{X > u\} du + \int_2^3 P\{X > u\} du + \dots$ But, X takes on only integer values. Thus, for any given nonnegative integer k , it must be that for all $u \in [k, k+1)$, $P\{X > u\} = P\{X > k\}$. Hence, $\int_k^{k+1} P\{X > u\} du = P\{X > k\}$ and thus $E[X] = P\{X > 0\} + P\{X > 1\} + \dots = \sum_{k=0}^{\infty} P\{X > k\}$.

This is the CompE version: some spoilsport EEs insist on writing this result as $E[X] = \sum_{k=1}^{\infty} P\{X \geq k\}$.

(b) If X is a geometric random variable with parameter p , then $P\{X > k\} = P\{\text{first } k \text{ trials ended in failure}\} = (1-p)^k$. This formula holds even when $k = 0$, since obviously $P\{X > 0\} = 1$.

$$\text{Hence, } E[X] = \sum_{k=1}^{\infty} P\{X \geq k\} = \sum_{k=0}^{\infty} P\{X > k\} = \sum_{k=0}^{\infty} (1-p)^k = 1 + (1-p) + (1-p)^2 + \dots = \frac{1}{1-(1-p)} = \frac{1}{p}$$

4.(a) This is a valid pdf. (b) This is a valid pdf.

(c) This is not a valid pdf because the function is negative for $0 < u < 1$. However, $\int_0^1 \ln u \, du = u \ln u - u \Big|_0^1$

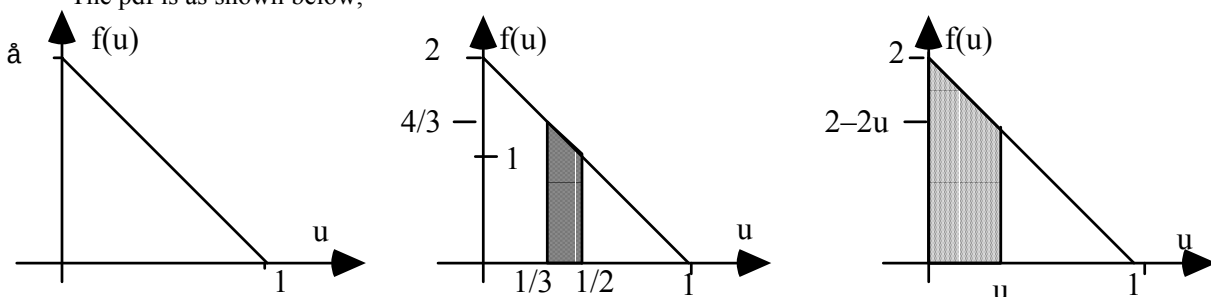
$= -1$, so $-\ln u$, $u \in (0, 1)$ is a valid pdf.

(d) This is not a valid pdf because the function is negative for $u \in (0, 1)$. Also, since the function is positive for $u \in (1, 2)$, $Cf(u)$ cannot be a valid pdf for any C .

(e) This is a valid pdf.

- (f) Although $\int_0^3 (2/3)(u-1) du = (1/3)(u-1)^2 \Big|_0^3 = 1$, this is not a valid pdf because the function is negative for $u \in (0,1)$. Also, since the function is positive for $u \in (1,3)$, $Cf(u)$ cannot be a valid pdf for any C .
- (g) Since $\int_0^\infty \exp(-2u) du = 1/2$, this is not a valid pdf. However, $2 \exp(-2u)$, $u \in (0, \infty)$ is a valid pdf; it is the exponential density with parameter 2.
- (h) Although $\int_0^\infty 4\exp(-2u) - \exp(-u) du = 2 - 1 = 1$, this is not a valid pdf because the function is negative for $u > \ln 4$. Since the function is positive for $u < \ln 4$, $Cf(u)$ cannot be a valid pdf for any C .

5. The pdf is as shown below,



The total area under the curve is $(1/2) \times \alpha \times 1 = \alpha/2$ which gives $\alpha = 2$.

- (a) $P\{6X^2 > 5X - 1\} = P\{6X^2 - 5X + 1 > 0\} = P\{(3X - 1)(2X - 1) > 0\}$
 $= P[(\{X > 1/3\} \cap \{X > 1/2\}) \cup (\{X < 1/3\} \cap \{X < 1/2\})] = P[\{X > 1/2\} \cup \{X < 1/3\}]$
 $= 1 - P[1/3 \leq X \leq 1/2]$. Now, $P[1/3 \leq X \leq 1/2] = \text{area under the pdf from } 1/3 \text{ to } 1/2$.
This is the difference of two triangular areas and is thus $(1/2) \times [(4/3) \times (2/3) - 1 \times (1/2)] = 7/36$.
Hence, $P\{6X^2 > 5X - 1\} = 1 - P[1/3 \leq X \leq 1/2] = 1 - 7/36 = 29/36$.
- (b) $F_X(u)$ is the area under the pdf upto the point u . By inspection, we have that $F_X(u) = 0$ if $u < 0$ and $F_X(u) = 1$ if $u > 1$. For $0 \leq u \leq 1$, $F_X(u)$ is the area shaded in the right-hand figure above
 $= 1 - (1/2) \times (1-u) \times (2-2u) = 1 - (1-u)^2 = 2u - u^2$.

6.(a) No, the tank is empty after 500 gallons have been sold, and towards the end of the week, the people asking for the extra 180 gallons are s.o.l.

(b) Sure, with 70 gallons to spare.

(c) The weekly demand can be satisfied if it (the demand) does not exceed 0.5. Thus,

$$P\{X \leq 0.5\} = \int_0^{0.5} 5(1-u)^4 du = -(1-u)^5 \Big|_0^{0.5} = 1 - \left(\frac{1}{2}\right)^5 = \frac{31}{32}$$

(d) We want the smallest value of C such that $P\{X > C\} \leq 10^{-5}$. But,

$$P\{X > C\} = \int_C^1 5(1-u)^4 du = -(1-u)^5 \Big|_C^1 = (1-C)^5 \leq 10^{-5} \text{ if } C \geq 0.9. \text{ Thus, a 900 gallon tank is}$$

required to achieve the desired goal.

(e,f,g) Let Y denote the amount of gasoline sold per week (measured in thousands of gallons). Then,

$Y = \begin{cases} X & \text{if } X \leq C, \\ C & \text{if } X > C. \end{cases}$ Hence, the weekly gross profit is $640Y$, and the **average** weekly **gross** profit is

$$E[640Y] = \int_0^C 5(1-u)^4 640u du + \int_C^1 5(1-u)^4 640C du = \frac{640}{6} [1 - (1-C)^6]. \text{ As a function of } C, \text{ the size of the}$$

tank, this increases from 0 if $C = 0$ (no tank!) to $\$640/6$ if $C = 1$. Now, the average **net** profit

$= E[640Y - 20C] = E[640Y] - 20C = (640/6)[1 - (1 - C)^6] - 20C$. This has a maximum if C satisfies $640(1 - C)^5 = 20$ i.e. $(1 - C)^5 = (1/32)$, i.e. if $C = 1/2$. Thus, a 500 gallon tank that costs \$10 per week to rent gives an average gross profit of \$630/6 and the maximum average net profit of \$570/6 per week. In contrast, a full-sized 1000 gallon tank gives only a slightly larger gross profit of \$640/6 per week, but a smaller net profit of \$520/6 per week. Of course, there may be intangible losses in goodwill if the tank is emptied before the week ends. However, the probability of this happening is only $1/32$ when $C = 1/2$.

$$7.(a) \quad E[Y] = E[X^2] = \int_{-1}^{+1} u^2(1/2)du = 1/3. \quad E[Y^2] = E[X^4] = \int_{-1}^{+1} u^4(1/2)du = 1/5.$$

Hence, $\text{var}(Y) = E[Y^2] - (E[Y])^2 = (1/5) - (1/3)^2 = 4/45$.

$$(b) \quad E[Z] = \int_{-\infty}^{\infty} g(u)f(u)du = \int_{-1}^0 -u^2(1/2)du + \int_0^1 u^2(1/2)du = -1/3 + 1/3 = 0.$$

$$(c) \quad Y = \sin(\pi X/2) = \begin{cases} 0, & \text{if } X \text{ is even,} \\ +1, & \text{if } X = 4k+1, k = 0, 1, \dots \\ -1, & \text{if } X = 4k+3, k = 0, 1, \dots \end{cases} \quad \text{and hence by LOTUS,}$$

$$E[Y] = \sum_{k=1}^{\infty} \sin(\pi k/2) \cdot P\{X = k\} = \frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \frac{1}{128} + \dots = \frac{1}{2} \cdot [1 - x + x^2 - x^3 + \dots] \quad \text{where } x = \frac{1}{4}$$

which, by the binomial theorem, is $\frac{1}{2} \frac{1}{1 + (1/4)} = \frac{2}{5}$

$$\text{Otherwise,} \quad p_Y(0) = \sum_{k=1}^{\infty} P\{X = 2k\} = \sum_{k=1}^{\infty} (1/2)^{2k} = (1/4) \sum_{k=0}^{\infty} (1/4)^k = 1/3$$

$$p_Y(1) = \sum_{k=0}^{\infty} P\{X = 4k+1\} = \sum_{k=0}^{\infty} (1/2)^{4k+1} = (1/2) \sum_{k=0}^{\infty} (1/16)^k = 8/15$$

$$p_Y(-1) = \sum_{k=0}^{\infty} P\{X = 4k+3\} = \sum_{k=0}^{\infty} (1/2)^{4k+3} = (1/8) \sum_{k=0}^{\infty} (1/16)^k = 2/15$$

giving $E[Y] = (-1) \cdot (2/15) + 0 \cdot (1/3) + (1) \cdot (8/15) = 2/5$.

8.(a) Since X takes on values between 0 and 1, so does Y and thus $\alpha = 0$ and $\beta = 1$.

Obviously, $P\{Y \leq \alpha - 1\} = P\{Y \leq -1\} = 0$ and $P\{Y > 2\beta\} = P\{Y > 2\} = 0$.

(b) Let $0 \leq v \leq 1$. Then, $F_Y(v) = P\{Y \leq v\} = P\{(1 - X)^2 \leq v\} = P\{-\sqrt{v} \leq 1 - X \leq \sqrt{v}\} = P\{X \geq 1 - \sqrt{v}\}$
 $= 1 - F_X(1 - \sqrt{v}) = (1 - (1 - \sqrt{v}))^2 = v$ where we used the result that $F_X(u) = 1 - (1 - u)^2$.

$$\text{Hence, } F_Y(v) = \begin{cases} 0, & u < 0, \\ v, & 0 \leq v \leq 1, \\ 1, & v > 1. \end{cases}$$

(c) A sketch of the function $F_Y(v)$ reveals that it is a nondecreasing continuous function. It is not differentiable at $\alpha = 0$ or at $\beta = 1$.

(d) Yes, we get the same values for $P\{Y \leq \alpha - 1\}$ and $P\{Y > 2\beta\}$ as in part (a).

$$9.(a) \quad E[R] = \int_0^1 3\rho^3 d\rho = \frac{3}{4}; \quad E[V] = E[4\pi R^3/3] = \int_0^1 4\pi \rho^5 d\rho = \frac{2\pi}{3}; \quad E[A] = E[4\pi R^2] = \int_0^1 12\pi \rho^4 d\rho = \frac{12\pi}{5}.$$

The average volume $E[V] = E[4\pi R^3/3]$ corresponds to a sphere of radius $(1/2)^{1/3}$ and the average surface area $E[A] = E[4\pi R^2]$ to a sphere of radius $(3/5)^{1/2}$. Note that $E[V] = E[4\pi R^3/3] \neq 4\pi(E[R])^3/3$, etc. This illustrates the general result that $E[g(X)]$ hardly ever equals $g(E[X])$. You will save yourself a lot of grief if you keep this in mind: that $E[g(X)] = g(E[X])$ is a common misconception among the instochaste.

(b) The volume V has values in the range $(0, 4\pi/3)$. For any u , $0 < u < 4\pi/3$, $F_V(u) = P\{V \leq u\}$

$$= P\{4\pi R^3/3 \leq u\} = P\{R \leq \sqrt[3]{3u/4\pi}\} = F_R(\sqrt[3]{3u/4\pi}) = 3u/4\pi \quad \text{since } F_R(\rho) = \rho^3 \text{ for } 0 < \rho < 1. \text{ Hence, } f_V(u) \text{ is uniform on } (0, 4\pi/3).$$

(c) Obviously, $E[V] = \text{midpoint of uniform pdf} = 2\pi/3$ as in part (a)

- (d) The electrical charge is uniformly distributed on the surface of the sphere. The surface charge density is $S = Q/4\pi R^2 > Q/4\pi$. For $x > Q/4\pi$,

$$F_S(x) = P\{S \leq x\} = P\{Q/4\pi R^2 \leq x\} = P\{1 > R \sqrt{Q/4\pi x}\} = 1 - (Q/4\pi x)^{1.5}.$$

Hence, $f_S(x) = (3/2x)(Q/4\pi x)^{1.5}$ for $x > Q/4\pi$, and 0 otherwise.

- 10.(a) If $X \geq H$, he sells all the papers. This event occurs with probability $P\{X \geq H\} = 1 - F_X(H^-) = 1 - F_X(H-1)$. (Why on earth does $F_X(H^-)$ equal $F_X(H-1)$?)
- (b) If demand $X \leq H$, the newsboy sells X papers and returns $H-X$ papers to the publishers. Thus, his profit is $(c_3 - c_2)X - (c_2 - c_1)(H-X) = (c_3 - c_1)X - (c_2 - c_1)H$. Otherwise, $X > H$, and the profit is $(c_3 - c_2)H$. Hence, Z is related to X and H as

$$Z = \begin{cases} (c_3 - c_1)X - (c_2 - c_1)H, & 0 \leq X \leq H, \\ (c_3 - c_2)H, & X > H. \end{cases}$$

- (c) LOTUS gives $E[Z] = \sum_{k=0}^H p_X(k)[(c_3 - c_1)k - (c_2 - c_1)H] + \sum_{k=H+1}^{\infty} p_X(k)(c_3 - c_2)H = g(H)$. Note that we can factor out $(c_3 - c_2)H$ from the second sum and simplify it to $(c_3 - c_2)HP\{X \geq H+1\} = (c_3 - c_2)H(1 - F_X(H))$.
- (d) The extra $((H+1)$ -th) paper provides extra profit of $(c_3 - c_2)$ cents if $X > H$ (which occurs with probability $1 - F_X(H)$) and a loss of $(c_2 - c_1)$ cents if $X \leq H$ (which occurs with probability $F_X(H)$). Hence, the average **additional** profit when one extra paper (over and above the usual number H) is purchased is given by $A(H) = (c_3 - c_2)(1 - F_X(H)) - (c_2 - c_1)F_X(H) = (c_3 - c_2) - (c_3 - c_1)F_X(H)$.
- (e) The CDF is a non-decreasing function, i.e. ... $F_X(H-1) \leq F_X(H) \leq F_X(H+1)$..., and hence $A(H)$ is a non-increasing function of H , i.e. ... $A(H-1) \geq A(H) \geq A(H+1)$...
- (f) $A(H) = (c_3 - c_2) - (c_3 - c_1)F_X(H) \leq 0$ if $F_X(H) \geq (c_3 - c_2)/(c_3 - c_1)$. But, $c_3 - c_1 > c_3 - c_2 > 0$, that is, $0 < (c_3 - c_2)/(c_3 - c_1) < 1$ so that as H increases, $F_X(H)$ eventually exceeds the quantity $(c_3 - c_2)/(c_3 - c_1)$ as it increases towards its limiting value $F_X(\infty) = 1$.
- (g) Extra papers mean extra profits as long as $A(H) > 0$. Since $A(H)$ is negative for large H , let H_0 denote the smallest integer for which $A(H_0) \leq 0$, i.e. H_0 is the smallest integer for which $F_X(H_0) \geq (c_3 - c_2)/(c_3 - c_1)$. This is the number of papers that should be bought. (If $F_X(H_0) = (c_3 - c_2)/(c_3 - c_1)$ exactly, the additional average profit from the (H_0+1) -th paper is 0. Why work harder than you have to?)