

5. Let $A_2, A_3, \dots, A_{12}$ denote the events that 2, 3, 4, $\dots$, 12 is rolled with two fair dice.
- (a) $P(\text{winning on first roll}) = P(A_7 \cup A_{11}) = P(A_7) + P(A_{11}) = 6/36 + 2/36 = 2/9$.
 $P(\text{losing on the first roll}) = P(A_2 \cup A_3 \cup A_{12}) = 1/36 + 2/36 + 1/36 = 1/9$.
- (b) $P(\text{shooter's point is 4}) = P(A_4) = 3/36 = 1/12 = P(A_{10}) = P(\text{shooter's point is 10})$.
 $P(\text{shooter's point is 5}) = P(A_5) = 4/36 = 1/9 = P(A_9) = P(\text{shooter's point is 9})$.
 $P(\text{shooter's point is 6}) = P(A_6) = 5/36 = P(A_8) = P(\text{shooter's point is 8})$.
- (c) For disjoint events A and B, $P(\text{event A occurs before B}) = P(A)/(P(A) + P(B))$. Hence,
 $P(\text{shooter makes point} | \text{shooter's point is 4}) = 3/(3+6) = 1/3 = P(\text{shooter makes point} | \text{shooter's point is 10})$
 $P(\text{shooter makes point} | \text{shooter's point is 5}) = 4/(4+6) = 2/5 = P(\text{shooter makes point} | \text{shooter's point is 9})$
 $P(\text{shooter makes point} | \text{shooter's point is 6}) = 5/(5+6) = 5/11 = P(\text{shooter makes point} | \text{shooter's point is 8})$
- (d) $P(\text{winning at craps}) = P(\text{winning on first roll}) + \sum_i P(\text{making point} | \text{point is } i) P(\text{shooter's point is } i)$
 $= (8/36) + 2 \times [(3/36) \times (1/3) + (4/36) \times (2/5) + (5/36) \times (5/11)] = (8/36) + (2/36) \times [1 + 8/5 + 25/11]$
 $= (8/36) + (2/36) \times [55 + 88 + 125]/55 = (440 + 536)/(36 \times 55) = 244/495 = 0.49292929\dots$
- (e) $P(\text{making 8 the hard way}) = P(\text{rolling 4-4 before rolling 7 or rolling 8 the easy way})$
 $= (1/36)/(1/36 + (6/36 + 4/36)) = 1/11$. Thus, in a long sequence of trials when the shooter's point is 8, one-eleventh of the time you win \$10 while ten-eleventh of the time you lose your \$1. Thus, in the long run, you neither make nor lose money on the average, and the bet is perfectly fair.
- 6.(a) The game winner wins the last point in the game, i.e., a score of 4-x or x-4 (with $0 \leq x < 3$) means that the x points won by the loser must have been among the first 3+x points. Thus,
 $P(4-x) = \binom{3+x}{x} p^3 q^x$, $P(x-4) = \binom{3+x}{x} q^3 p^x$, giving $P(4-0) = p^4$; $P(4-1) = 4p^4q$; $P(4-2) = 10p^4q^2$;
 $P(0-4) = q^4$; $P(1-4) = 4pq^4$; $P(2-4) = 10p^2q^4$.
- (b) Deuce is reached with probability $\binom{6}{3} p^3 q^3 = 20p^3 q^3$. To show that the sum is 1, just multiply it all out and hope you don't make an error (or use Mathematica).
- (c) $P(A \text{ wins next two points} | \text{deuce}) = p^2$. $P(B \text{ wins next two points} | \text{deuce}) = q^2$.
 $P(\text{each wins one point} | \text{deuce}) = P(\text{deuce again}) = 2pq$ (why 2?)
- (d) $P(A \text{ wins the game} | \text{deuce}) = p^2 + (2pq)p^2 + (2pq)^2 p^2 + \dots = p^2/(1 - 2pq)$. Similarly,
 $P(B \text{ wins the game} | \text{deuce}) = q^2 + (2pq)q^2 + (2pq)^2 q^2 + \dots = q^2/(1 - 2pq)$. As a sanity check, we note that
 $1 = (p+q)^2 = p^2 + q^2 + 2pq$ so that $1 - 2pq = p^2 + q^2$ implying that $p^2/(1 - 2pq) + q^2/(1 - 2pq) = 1$, i.e. one of A and B *does* ultimately win the game.
- 7.(a) The bill passes if the Conservative Republicans and at least one other group votes for it, i.e. the event $A \cap (B \cup C \cup D)$ occurs, or if all groups except the Conservative Republicans vote for it, i.e. the event $A^c \cap B \cap C \cap D$ occurs. Note that these are disjoint events (Why?). Now, $(B \cup C \cup D)^c = B^c \cap C^c \cap D^c$ by DeMorgan's theorem, and thus we can partition A into two disjoint events as follows:
 $A = A \cap (B \cup C \cup D) \cup A \cap B^c \cap C^c \cap D^c$. Hence, we see that
 $P(A \cap (B \cup C \cup D)) = P(A) - P(A \cap B^c \cap C^c \cap D^c) = P(A) - P(A)P(B^c)P(C^c)P(D^c)$ by independence. Thus,
 $P(\text{bill passes}) = P(A \cap (B \cup C \cup D)) + P(A^c \cap B \cap C \cap D) = P(A) - P(A)P(B^c)P(C^c)P(D^c) +$
 $P(A^c)P(B)P(C)P(D)$ (independence again!) $= 0.9 \times [1 - 0.4 \times 0.5 \times 0.8] + 0.1 \times 0.6 \times 0.5 \times 0.2 = 0.756 + 0.006 = 0.762$.
- (b) The Conservative Republicans must be in favor for the motion to override to pass. Furthermore, since politics makes strange bedfellows, either the Liberal Democrats must also be in favor or both the other groups must also be in favor. Hence,
 $P(\text{motion to override passes}) = P[E(H \cup FG)] = P(E) \times P(H \cup FG)$ by independence of E and $H \cup FG$
 $= P(E) \times [P(H) + P(FG) - P(FGH)] = P(E) \times [P(H) + P(F)P(G) - P(F)P(G)P(H)]$ (by independence again!)
 $= 0.99 \times [0.1 + 0.4 \times 0.6 - 0.6 \times 0.4 \times 0.1] = 0.31284$.
- 6.(a) With N-ary replication of each component, each replicated component has probability of failure p^N since all the N copies of the component must fail for the replicated component to fail. The probability that all M replicated components are in working order is thus $(1 - p^N)^M$, and the probability of failure for the system

in the left-hand figure is thus $1 - (1 - p^N)^M$

On the other hand, suppose that the entire system is replicated N times as shown in the right-hand figure.

Since each copy of the entire system fails with probability $1 - (1 - p)^M$, the probability of failure for the system in the right-hand figure is $[1 - (1 - p)^M]^N$.

- (b) $1 - (1 - 0.2^N)^5 \leq 0.001 \Rightarrow (1 - 0.2^N)^5 \geq 0.999 \Rightarrow (1 - 0.2^N) \geq 0.99979992 \Rightarrow 0.2^N \leq 0.000200080$
 $\Rightarrow N \geq 6$. More generally, $N \geq \frac{\ln(1 - (1 - \alpha)^{1/M})}{\ln p}$ where α is the desired maximum failure probability.

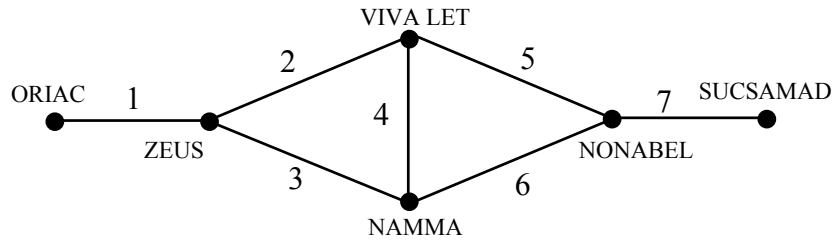
$$[1 - (1 - 0.2)^5]^N \leq 0.001 \Rightarrow 0.67232^N \leq 0.001 \Rightarrow N \geq 18. \text{ More generally, } N \geq \frac{\ln \alpha}{\ln(1 - (1 - p)^M)}$$

Put another way, in our car example, having a single gigantic car with 6 engines, 6 transmissions, 6 brakes, ... will provide more reliable transportation than having 17 separate cars. This may (or may not!) explain why people buy one large SUV rather than 18 compact cars!

- (c) If $M = 1000$, we get $N \geq 9$ in the one case. The other case blows up on many calculators because the quantity $1 - (1 - p)^M \approx 1 - \exp(-Mp)$ (huh?) $= 1 - \exp(-200)$ is very nearly 1. The Unix high-precision utility *bc* gives its value as 0.999...99861610344 where there are 87 9's preceding the 8616.... Since most cars have 1000 (or more!) parts, the number of separate cars required to provide reliable transportation is far larger than the number of cars in this world! On the other hand, a car with nine engines, nine transmissions etc will provide the necessary reliability but it will be difficult to squeeze into most parking spaces!

Moral: Replicate at the component level, instead of replicating entire systems. It's a lot cheaper!

9. Linguistic innocents are reminded that both Arabic and Hebrew are read from right to left. Number the links as shown. Let V_i denote the event that the i -th link is viable (i.e. working) and let A denote the event that it is possible to call from ORIAC to SUCSAMAD.



- (a) If V_4^c has occurred, i.e. the VIVA LET to NAMMA link has been severed, then,
 $P(A|V_4^c) = P(V_1 V_2 V_5 V_7 \cup V_1 V_3 V_6 V_7 | V_4^c) = P(V_1 V_2 V_5 V_7 \cup V_1 V_3 V_6 V_7)$ by independence!
 $= P(V_1 V_2 V_5 V_7) + P(V_1 V_3 V_6 V_7) - P(V_1 V_2 V_3 V_5 V_6 V_7) = 2(1 - p)^4 - (1 - p)^6$ by independence! On the other hand, if V_4 has occurred, i.e. the VIVA LET to NAMMA link is viable, then
 $P(A|V_4) = P(V_1 (V_2 \cup V_3) (V_5 \cup V_6) V_7 | V_4) = P(V_1 (V_2 \cup V_3) (V_5 \cup V_6) V_7)$ by independence!
 $= P(V_1) \times P(V_2 \cup V_3) \times P(V_5 \cup V_6) \times P(V_7) = (1 - p)^2 \times [(1 - p) + (1 - p) - (1 - p)^2]^2 = (1 - p)^2 (1 - p^2)^2$
 $= (1 - p)^4 (1 + p)^2$ by repeated use of the independence of events. Hence,
 $P(A) = p \times [2(1 - p)^4 - (1 - p)^6] + (1 - p) \times (1 - p)^4 (1 + p)^2 = (1 - p)^4 \times (1 + 2p + p^2 - 2p^3)$
- (b) $P(V_3^c | A) = P(A \cap V_3^c) / P(A)$. But, $P(A \cap V_3^c) = P[V_1 V_2 (V_5 \cup V_6) V_7 V_3^c]$
 $= \{(1 - p)^2 \times [(1 - p) + (1 - p)^2 - (1 - p)^3] \times (1 - p) \times p = (1 - p)^4 \times (1 + p - p^2) \times p$. Hence,
 $P(V_3 | A) = 1 - P(V_3^c | A) = 1 - \frac{(1 - p)^4 \times (1 + p - p^2) \times p}{(1 - p)^4 \times (1 + 2p + p^2 - 2p^3)} = \frac{(1 + p - p^3)}{1 + 2p + p^2 - 2p^3}$.
- (c) Clearly X is nonnegative, and cannot exceed 70. Also, $X = 0$ if either link #1 or link #7 has been severed i.e. if $V_1^c \cup V_7^c$ occurs. Thus, assume that $V_1 V_7$ has occurred. As in part(a), we can calculate the capacity given that V_4^c has occurred and also given that V_4 has occurred. Thus, given that $V_1 V_4^c V_7$ has occurred, the capacity from ORIAC to SUCSAMAD is 20 if only one of the paths from ZEUS to NONABEL is working, and 40 if both are working, etc.. These capacities are indicated on the two "Karnaugh" maps shown on the next page.

		$V_1 V_4^c V_7$ occurred			
$V_2 V_5 \rightarrow$		00	01	11	10
$V_3 V_6 \downarrow$	00	0	0	20	0
	01	0	0	20	0
	11	20	20	40	20
	10	0	0	20	0

		V₁V₄V₇ occurred			
V ₂ V ₃ →		00	01	11	10
V ₅ V ₆ ↓	00	0	0	0	0
	01	0	20	20	20
	11	0	50	70	20
	10	0	50	50	20

From this, we get $P\{X = 70\} = (1-p)^7$ {**check:** all links must work to get a capacity of 70},
 $P\{X = 50\} = [(1-p)^2 - (1-p)^4](1-p)^3 = (1-p)^5 - (1-p)^7 = (1-p)^5[2p - p^2]$ {**check:** links 1, 3, 4, 5, 7 must work, and at least one of 2 and 6 must have failed which has prob $p + p - p^2$, **why?**},
 $P\{X = 40\} = (1-p)^6 p$ {**check:** all links except 4 must be in working condition},
 $P\{X = 20\} = 2(1-p)^4 p^3 + 7(1-p)^5 p^2 + 2(1-p)^6 p$, {**check:** the first term corresponds to 1-2-5-7 and 1-3-6-7 as the only working links, the last term corresponds to 3 or 5 being the only failed link, and the middle term corresponds to everything working **except** (2 and exactly one of 4 and 5) or (3 and exactly one of 4, 5, 6) or (4 and exactly one of 5 and 6)},
 $P\{X = 0\} = 1 - P\{X \neq 0\} = 1 - [(1-p)^7 + (1-p)^5 - (1-p)^7 + p(1-p)^6 + 2(1-p)^4 p^3 + 7(1-p)^5 p^2 + 2(1-p)^6 p]$
 $= 1 - (1-p)^4(1 + 2p + p^2 - 2p^3) = 1 - P(A)$ as it should be!
 Finally, $E[X] = 70P\{X = 70\} + 50P\{X = 50\} + 40P\{X = 40\} + 20P\{X = 20\}$
 $= 20(1-p)^7 + 80(1-p)^6 p + 50(1-p)^5 + 40(1-p)^4 p^3 + 140(1-p)^5 p^2$. As a quick check, note that $E[X] = 70$ if $p = 0$ (totally incompetent terrorists!) and $E[X] = 0$ if $p = 1$ (fearless liberation fighters not afraid to die in the attempt!). It is interesting to plot these probabilities and expected values as a function of p . $P(A) \approx 80\%$ for $p = 0.05$ and decreases to 50% at roughly $p = 0.2419\dots$ when $E[X] = 21\dots$

- 10.(a) $P(3 \text{ or more failures}) = \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + p^5 = 10p^3 - 15p^4 + 6p^5$. Or, using the hint, if both IV and V have failed (this has probability p^2 of occurring), then output is wrong if at least one of I, II, and III have failed. This has probability $1 - (1-p)^3$ since $(1-p)^3$ is the probability that I, II, and III are all working correctly. If exactly one of IV and V have failed (this has probability $2p(1-p)$ of occurring), then at least two of I, II, and III must fail for the output to be incorrect. This has probability $3p^2 - 2p^3$ as discussed in class. If both IV and V are working (with probability $(1-p)^2$), then output is incorrect if all three of I, II, and III have failed which has probability p^3 . Hence,
 $P(\text{incorrect}) = p^2[1 - (1-p)^3] + 2p(1-p)[3p^2 - 2p^3] + (1-p)^2 p^3 = 10p^3 - 15p^4 + 6p^5$ once again. Note that the dominant term is $10p^3$ whereas in the TMR system, it was $3p^2$. Thus, QMR improves reliability even more than TMR (but it also costs more). TANSTAAFL!
- (b) As long as any three circuits are working, the QMR system output is correct. So, we have paths with three links from In to Out, and there are $\binom{5}{3} = \binom{5}{2} = \frac{5}{1} \times \frac{4}{2} = 10$ such paths in the graph model of the QMR system. Each circuit is represented by links in $\binom{4}{2} = 6$ paths, e.g. if V fails, we are left with 4 viable paths I-II-III, I-II-IV, I-III-IV, and II-III-IV, and failure of another one circuit leaves just one path.