

1. Let B_i and R_i respectively denote the events that the i -th ball is black or red. Then,

$$P(B_1) = \frac{b}{b+r} \text{ and } P(R_1) = \frac{r}{b+r}. \text{ Now, } P(R_2) = P(R_2|R_1)P(R_1) + P(R_2|B_1)P(B_1) = \frac{r+c}{b+r+c} \times \frac{r}{b+r} + \frac{r}{b+r+c} \times \frac{b}{b+r} = \frac{r}{b+r} = P(R_1)! \text{ Hence, it must also be that } P(B_2) = 1 - P(R_2) = \frac{b}{b+r} = P(B_1). \text{ Furthermore,}$$

$$P(B_1|R_2) = P(R_2|B_1)P(B_1)/P(R_2) = \frac{r}{b+r+c} \times \frac{b}{b+r} / \frac{r}{b+r} = \frac{b}{b+r+c} \text{ as stated in the problem. Note also that}$$

$$P(R_3) = P(R_3R_2R_1) + P(R_3R_2B_1) + P(R_3B_2R_1) + P(R_3B_2B_1) \quad (\text{why?})$$

$$= P(R_3|R_2R_1)P(R_2|R_1)P(R_1) + P(R_3|R_2B_1)P(R_2|B_1)P(B_1) + P(R_3|B_2R_1)P(B_2|R_1)P(R_1) \quad (\text{why?})$$

$$+ P(R_3|B_2B_1)P(B_2|B_1)P(B_1) = r/(b+r) \text{ also! (Verify this).}$$

- 2.(a) Obviously United Airlines has better on-time performance at all five airports.

(b) $P(T|UC)P(C|U) = \frac{P(TUC)}{P(UC)} \times \frac{P(UC)}{P(U)} = \frac{P(TUC)}{P(U)}$ and similarly for the other terms. Hence, the right side of the given expression is $\frac{P(TUC) + P(TUL) + P(TUX) + P(TUD) + P(TUF)}{P(U)}$.

But, $(TUC) \cup (TUL) \cup (TUX) \cup (TUD) \cup (TUF)$ is a partition of the set TU , and thus the numerator above is just $P(TU)$, and the ratio is, by definition, $P(T|U)$. Also,

- (c) $P(T|W) = P(T|WC)P(C|W) + P(T|WL)P(L|W) + P(T|WX)P(X|W) + P(T|WD)P(D|W) + P(T|WF)P(F|W)$
Plugging and chugging, $P(T|U) = 0.8655 < 0.896 = P(T|W)$. The reason for the discrepancy is that America West has most of its flights into sunny Phoenix where flights have a good chance of being on time. In contrast, United has only a few flights to Phoenix and has lots of flights into snowy Chicago (and foggy San Francisco.) Note that 0.69 of the 0.896 of $P(T|W)$ comes from Phoenix, whereas United gets only 0.0475 of the 0.8655 of $P(T|U)$ from Phoenix (where it performs the best).

The moral for the marketing departments is that America West should advertise itself as most-often-on-time, whereas United should advertise itself as most-often-on-time-where-you-wanna-go.

Comment: This problem is based on a real-life case. The University of California at Berkeley (of all places!) was accused by the EEOC of discriminating against women applying to graduate school. Each Department actually admitted a larger fraction of its women applicants than of its men applicants, but the overall statistics showed a smaller fraction of women applicants being admitted as compared to men applicants.

- 3.(a) Two boxes, of course!

(b) For $n \geq 2$, $P\{X = n\} = P\{LLL \dots LD\} + P\{DDD \dots DL\} = (2/3)^{n-1}(1/3) + (1/3)^{n-1}(2/3) = (2^{n-1}+2)/3^n$.

$$\sum_{n=2}^{\infty} (2^{n-1}+2)/3^n = \frac{2}{9} \cdot \left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots \right] + \frac{2}{9} \cdot \left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \dots \right] = \frac{2}{9} \left[\frac{1}{1-2/3} + \frac{1}{1-1/3} \right] = 1.$$

- (c) If the first box contains a picture of Luke, then Jimmy has to wait for a box with Darth. Thus, the number of *additional* boxes is a geometric random variable Y with parameter $1/3$. Conversely, if the first box contains a picture of Darth, then Jimmy has to wait for a box with Luke. Thus, the number of *additional* boxes is a geometric random variable Z with parameter $2/3$. Thus, for $n \geq 2$,

$$P\{X = n \mid \text{first box is L}\} = P\{Y = n-1\} = (2/3)^{n-2}(1/3) \quad (\text{note that } X = 1 + Y)$$

$$\text{and } P\{X = n \mid \text{first box is D}\} = P\{Z = n-1\} = (1/3)^{n-2}(2/3). \quad (\text{note that } X = 1 + Z)$$

We note for later use that the conditional mean and variance of X are given by

$$E[X \mid L] = E[1+Y] = 1 + E[Y] = 1 + 1/(1/3) = 4; \text{ var}(X \mid L) = \text{var}(1+Y) = \text{var}(Y) = (1 - 1/3)/(1/3)^2 = 6; E[X \mid D] = E[1+Z] = 1 + E[Z] = 1 + 1/(2/3) = 5/2; \text{ var}(X \mid D) = \text{var}(1+Z) = \text{var}(Z) = (1 - 2/3)/(2/3)^2 = 3/4.$$

- (d) $P\{X = n\} = P\{X = n \mid \text{first box is L}\}P\{\text{first box is L}\} + P\{X = n \mid \text{first box is D}\}P\{\text{first box is D}\}$
 $= (2/3)^{n-2}(1/3)(2/3) + (1/3)^{n-2}(2/3)(1/3) = (2^{n-1}+2)/3^n$ just as in part (b).

- (e) $E[X] = E[X \mid \text{first box is L}] \cdot P\{\text{first box is L}\} + E[X \mid \text{first box is D}] \cdot P\{\text{first box is D}\}$
 $= 4 \cdot (2/3) + (5/2) \cdot (1/3) = 7/2$. Note that the unconditional mean is between the two conditional means...
 $\text{var}(X)$ is the sum of the mean of the conditional variance $(6 \cdot (2/3) + (3/4) \cdot (1/3))$ and the variance of the conditional mean which is readily calculated as $(4^2 \cdot (2/3) + (5/2)^2 \cdot (1/3)) - (7/2)^2$ (huh? what formula is that?) or as $(4 - 7/2)^2 \cdot (2/3) + (5/2 - 7/2)^2 \cdot (1/3)$ giving us that $\text{var}(X) = 4.75$.

4. There are $\binom{11}{2} = \frac{11 \times 10}{1 \times 2} = 55$ pairs of letters that might fall off.

The 55 pairs can be classified into the pair OO, 5 *other* pairs from the set {H, O, O, N}, $4 \times 7 = 28$ pairs consisting of one letter from {H, O, O, N} and one from {C, A, T, T, A, G, A}, 3 pairs AA and 1 pair TT, and 17 *other* pairs from {C, A, T, T, A, G, A}.

If OO fell off, the sign will always read correctly.

If one of 5 other pairs from {H, O, O, N} fell off, the letters will always look right side up, but will be interchanged with probability 1/2.

For the 28 pairs consisting of one letter from {H, O, O, N} and one from {C, A, T, T, A, G, A}, the restored sign is correct with probability 1/4, incorrect but readable with probability 1/4, and one letter appears inverted with probability 1/2.

For the AA and TT pairs, the sign is correct with probability 1/4, has one letter inverted with probability 1/2 and has both letters inverted with probability 1/4.

The remaining 17 pairs are put back correctly with probability 1/8, interchanged but upright with probability 1/8, have one letter upside down with probability 1/2, and have both letters upside down with probability 1/4. This gives

$$(a) \quad P(\text{sign reads CHATTANOOGA}) = \left[1 + 5 \times \frac{1}{2} + 28 \times \frac{1}{4} + 4 \times \frac{1}{4} + 17 \times \frac{1}{8} \right] \times \frac{1}{55} = \frac{109}{440} \approx 25\%. \text{ Not Bad!}$$

$$(b) \quad P(\text{sign readable but not CHATTANOOGA}) = \left[5 \times \frac{1}{2} + 28 \times \frac{1}{4} + 17 \times \frac{1}{8} \right] \times \frac{1}{55} = \frac{93}{440}$$

$$(c) \quad P(\text{one letter seems to be upside down}) = \left[28 \times \frac{1}{2} + 4 \times \frac{1}{2} + 17 \times \frac{1}{2} \right] \times \frac{1}{55} = \frac{196}{440} = \frac{49}{110}$$

$$(d) \quad P(\text{two letters seem to be upside down}) = \left[4 \times \frac{1}{4} + 17 \times \frac{1}{4} \right] \times \frac{1}{55} = \frac{42}{440} = \frac{21}{220}$$

Sanity check: The sum of the probabilities is $(109+93+196+42)/440 = 1$.

(e) All letters seem to be right side up includes the case when the sign reads CHATTANOOGA! Now,

$$P(\text{at least one vowel} | \text{all seem to be right side up}) = \frac{P(\text{all seem to be right side up} \cap \text{at least one vowel})}{P(\text{all seem to be right side up})}$$

We know from (a) and (b) that the denominator is $(109+93)/440 = 101/220$.

Now, our list above can be further classified into

1 pair OO and 2 pairs each of the forms HO and NO (for these the letters always appear to be right side up),

14 of the form O and one from {CATTAGA} and 3 each of the form HA and NA (for these, the probability that the letters seem right side up is 1/2),

3 of the form AA, and 3 each of the form CA and GA, and 6 of the form TA (for these the probability that the letters seem to be right side up is 1/4). All other pairs {from the set CHTTNG} do not include a vowel.

Check: $P(\text{at least one vowel}) = 1 - P(\text{no vowel}) = 1 - [(6 \times 5)/(1 \times 2)]/55 = 40/55$
 $= (1 + 2 + 2 + 14 + 3 + 3 + 3 + 3 + 6)/55$ so we got 'em all!

From all this, we can readily compute $P(\text{all seem to be right side up} \cap \text{at least one vowel})$

$$= [(1+2+2) \times 1 + (14 + 3 + 3) \times 1/2 + (3 + 3 + 3 + 6) \times (1/4)]/55 = 75/220.$$

$$\text{Hence, } P(\text{at least one vowel} | \text{all seem to be right side up}) = \frac{75}{220} / \frac{101}{220} = \frac{75}{101}.$$

(f) The designated driver can correctly identify the letters that fell down if and only if the driver can see that two letters are obviously switched (possibly being turned upside down) or if two letters both appear to be upside down but in their correct places. Note that these are disjoint possibilities. For the various pairs of different types, we see that if OO fell down, the sign will still read CHATTANOOGA and the driver will not be able to identify with certainty which letters fell down.. If one of the 5 other pairs from {H, O, O, N} fell down, the driver can identify them if and only if they have been switched (probability 1/2); whether they have been flipped over doesn't matter. For the 28 pairs consisting of one letter from {H, O, O, N} and one from

{C, A, T, T, A, G, A}, the two letters must be swapped in order to be identifiable (probability 1/2), while for the 3 + 1 pairs AA and TT, both letters must be upside down to be identifiable with certainty. Finally, the remaining 17 *other* pairs from {C, A, T, T, A, G, A}, either both letters must be in the wrong place (probability 1/2) or they must be in the right place and both upside down (probability 1/8).

$$\text{Hence, } P\{\text{driver can identify correctly}\} = \left[\frac{1}{55} \times 0 + \frac{5}{55} \times \frac{1}{2} + \frac{28}{55} \times \frac{1}{2} + \frac{4}{55} \times \frac{1}{4} + \frac{17}{55} \times \frac{5}{8} \right] = \frac{225}{440} = \frac{45}{88} >$$

50%!!