

1. $P\{X \text{ is odd}\} = P\{X = 1\} + P\{X = 3\} + \dots + P\{X = M\}$ where $M = N$ if N is odd and $M = N - 1$ if N is even. Hence, $P\{X \text{ is odd}\} = \binom{N}{1} q^{N-1} p + \binom{N}{3} q^{N-3} p^3 + \dots + \binom{N}{M} q^{N-M} p^M$. Note that the last term is p^N if N is odd, and it is Nqp^{N-1} if N is even (whatever happened to $\binom{N}{M}$??)
- Now, $(q + p)^N = q^N + \binom{N}{1} q^{N-1} p + \binom{N}{2} q^{N-2} p^2 + \dots + \binom{N}{N-1} qp^{N-1} + p^N$ while
- $$(q - p)^N = q^N - \binom{N}{1} q^{N-1} p + \binom{N}{2} q^{N-2} p^2 - \dots + (-1)^{N-1} \binom{N}{N-1} qp^{N-1} + (-1)^N p^N.$$
- We subtract the second equation from the first one and note alternate terms cancel. Also if N is even, the last terms cancel, while if N is odd, the next-to-last terms cancel. In either case, we see that the result is just $2P\{X \text{ is odd}\}$, i.e. $P\{X \text{ is odd}\} = [(q+p)^N - (q-p)^N]/2 = [1 - (1-2p)^N]/2$.
2. $Y = 1, 2, 3$ passengers are left behind according as $X = 5, 6$, or 7 . Since X takes on values $5, 6, 7$ with probabilities $\frac{21}{128}, \frac{7}{128}, \frac{1}{128}$ respectively, we readily find that $E[Y] = \frac{1 \times 21 + 2 \times 7 + 3 \times 1}{128} = \frac{19}{64}$.
- 3.(a) A packet is received correctly if and only if all n bits are correct, which has probability $(1-p)^n = Q$. But, in case of errors, a re-transmission can occur (upto a maximum of 5 tries). Thus, $P\{\text{packet lost}\} = P\{5 \text{ unsuccessful transmissions}\} = (1-Q)^5$, and $P\{\text{packet transmission is successful}\} = 1 - (1-Q)^5$.
- (b) X_i takes on values $1, 2, 3, 4, 5$ with probabilities $Q, (1-Q) \cdot Q, (1-Q)^2 \cdot Q, (1-Q)^3 \cdot Q$, and $(1-Q)^4$ respectively. Note that although the first 4 probabilities are a geometric series, $P\{X = 5\}$ is *not* the probability that a geometric random variable has value 5! If the first four transmissions are not received correctly, then X is capped at 5, and $P\{X = 5\} = P\{\text{geometric random variable} > 4\} = (1-Q)^4$.
- (c) $E[X_i] = 1 \cdot Q + 2 \cdot (1-Q) \cdot Q + 3 \cdot (1-Q)^2 \cdot Q + 4 \cdot (1-Q)^3 \cdot Q + 5 \cdot (1-Q)^4$
- $$= Q \cdot [1 + (1-Q) + (1-Q)^2 + (1-Q)^3 + (1-Q)^4]$$
- $$+ Q \cdot [(1-Q) + (1-Q)^2 + (1-Q)^3 + (1-Q)^4]$$
- $$+ Q \cdot [(1-Q)^2 + (1-Q)^3 + (1-Q)^4]$$
- $$+ Q \cdot [(1-Q)^3 + (1-Q)^4]$$
- $$+ (1-Q)^4$$
- $$= Q \cdot \frac{1 - (1-Q)^4}{1 - (1-Q)} + (1-Q)^4 + Q \cdot (1-Q) \cdot \frac{1 - (1-Q)^3}{1 - (1-Q)} + (1-Q)^4$$
- $$+ Q \cdot (1-Q)^2 \cdot \frac{1 - (1-Q)^2}{1 - (1-Q)} + (1-Q)^4 + Q \cdot (1-Q)^3 + (1-Q)^4$$
- $$= 1 + (1-Q) + (1-Q)^2 + (1-Q)^3 + (1-Q)^4 = \frac{1 - (1-Q)^5}{1 - (1-Q)} = \frac{1 - (1-Q)^5}{Q}$$
- $$= \frac{1 - (1 - 5Q + 10Q^2 - 10Q^3 + 5Q^4 - Q^5)}{Q} = 5 - 10Q + 10Q^2 - 5Q^3 + Q^4$$
- (d) $P\{\text{all packets received successfully}\} = [1 - (1-Q)^5]^L$. Note that the answer is *not* Q^L , which is the probability that all packets are received successfully on the first attempt.
4. $(d/d\lambda)P\{X = k\} = (d/d\lambda)\exp(-\lambda)\lambda^k/k! = \exp(-\lambda)[k\lambda^{k-1}/k! - \lambda^k/k!] = 0$ at $\lambda = k$. Since $\exp(-\lambda)\lambda^k/k! > 0$ for all $\lambda \in (0, \infty)$ and has value 0 at $\lambda = 0$ and also at $\lambda = \infty$, it must have a maximum at $\lambda = k$. Thus the maximum-likelihood estimate of λ is $\hat{\lambda} = k$. Since $\lambda \approx Np$, this is consistent with the binomial estimate $\hat{p} = k/N$.
- 5.(a) Maximize your chances of living by predicting the most probable outcome. According to Proposition 6.1, p.145 in Ross) the most probable number of Heads is $\lfloor (N+1)p \rfloor = 110$, so predict that 110 Heads will occur. This has probability $\binom{1000}{110} (0.11)^{110} (0.89)^{890} = 0.04028\dots$ Not great odds of survival...
- It is important to understand that the answer to this problem is *not* the mean value Np , even though both formulas happen to give the same number in this particular instance.
- (b) $P(\text{Heads occurring for first time on } k\text{-th toss}) = (1-p)^{k-1}p$ is a *decreasing* function of k . So predict that Heads occurs for the first time on the first toss, and survive with probability 0.11! Note that if you had

chosen the mean value $1/p = 9.09\dots$ and rounded it off to 9, your chances of surviving would be only $(0.89)^8 0.11 = 0.0433\dots$. This is yet another illustration of the central idea: you have the best chance of survival if you predict that the most likely event will occur, and not if you predict that the “average” event will occur, i.e. use probabilistic notions correctly in order to live long and prosper!

- (c) $P(105\text{th head on } k\text{th toss}) = \binom{k-1}{104} p^{105} q^{k-105}$. To find which value of k maximizes this, consider the ratio $\frac{P(105\text{th head on } k\text{th toss})}{P(105\text{th head on } (k-1)\text{th toss})} = \frac{k-1}{k-105} q \geq 1$ as long as $k \leq \frac{105-q}{1-q} = 946.45\dots$. So predict that the 105th head will occur on the 946th toss. The probability of this event is $\binom{945}{104} p^{105} q^{841} = 0.00455\dots$ Ouch!
- (d) $P(\text{Heads occurring for first time on } 12\text{th toss}) = (1-p)^{11} p$. It is easy to show that this probability is maximum at $p^* = 1/12 = 0.08333\dots$
- (e) The maximum-likelihood estimate of p is the number p^* that maximizes $\binom{993}{299} p^{300} (1-p)^{694}$. As discussed in class, the maximum is at $p^* = 300/994 = 0.30181086519\dots$
- (f) The maximum-likelihood estimate of the unknown value of p is the number p^* that maximizes $\binom{1000}{300} p^{300} (1-p)^{700}$. We saw in class that the maximum is at $p^* = 300/1000 = 0.3$
- (g) The difference in the two cases is that after having observed the 300th head on the 994th toss, we can estimate p^* as $300/994 = 0.3018\dots$ *without knowing what happened on the next 6 trials*. If we are told that Tails occurred on the next 6 trials, we *do* receive additional information which causes us to be pessimistic about the value of p , and hence we revise the estimated value of p *downwards* to $300/1000 = 0.3$.

- 6.(a) $P(A|B^c) = 1 - P(A^c|B^c) = 0.6$. $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = 0.39$.
 $P(B|A) = P(AB)/P(A) = P(A|B)P(B)/P(A) = 0.21/0.39 = 7/13$.
- (b) $P(F) = P(EF)/P(E|F) = P(F|E)P(E)/P(E|F) = (1/2) - (1/4)/(1/3) = 3/8$.
- (c) $1 \geq P(G \cup H) = P(G) + P(H) - P(GH)$.
Hence, $P(GH) \geq 2/3 + 2/3 - 1 = 1/3$, and $P(G|H) = P(GH)/P(H) = P(GH)/(2/3) \geq (1/3)/(2/3) = 1/2$.
7. Given that $X = n$, we know that there was a success on the n -th trial, and $r-1$ successes in the previous $n-1$ trials so that $P\{X = n\} = p \cdot \binom{n-1}{r-1} p^{r-1} (1-p)^{(n-1)-(r-1)} = \binom{n-1}{r-1} p^r (1-p)^{n-r}$. If there is a success on the i -th trial also, then there are $r-2$ successes on the *other* $n-2$ trials preceding the n th, i.e.
 $P(\{X = n\} \cap \{\text{success on } i\text{-th}\}) = p \cdot p \cdot \binom{n-2}{r-2} p^{r-2} (1-p)^{(n-2)-(r-2)} = \binom{n-2}{r-2} p^r (1-p)^{n-r}$. Hence, the conditional probability is the ratio of the binomial coefficients which works out to $(r-1)/(n-1)$.
Alternatively, consider that given $X = n$, we know that there are $r-1$ successes in the previous $n-1$ trials. But, we saw in class that given that there are k successes on n trials, the probability of a success on the i -th trial is just k/n independent of p . So, here the answer must be $(r-1)/(n-1) \dots$

- 8.(a) $P\{X \leq 100\} = 1 - P\{X > 100\}$
 $= 1 - [P\{X = 105\} + P\{X = 104\} + P\{X = 103\} + P\{X = 102\} + P\{X = 101\}]$
 $= 1 -$
 $\left[(0.9)^{105} + \binom{105}{1} (0.9)^{104} (0.1) + \binom{105}{2} (0.9)^{103} (0.1)^2 + \binom{105}{3} (0.9)^{102} (0.1)^3 + \binom{105}{4} (0.9)^{101} (0.1)^4 \right]$
 $= 0.983283\dots$
- (b) If X is a binomial random variable with parameters (n, p) , then $Y = n - X$ is a binomial random variable with parameters $(n, 1-p)$. Thus, the number of no-shows is a binomial random variable with parameters $(105, 0.1)$. Since n is large and p is small, it is reasonable to approximate this as a Poisson random variable with parameter $\lambda = np = 10.5$.
- (c) $\{Y \geq 5\} = 1 - P\{Y < 5\} = 1 - [P\{Y = 0\} + P\{Y = 1\} + P\{Y = 2\} + P\{Y = 3\} + P\{Y = 4\}]$
 $= 1 - \exp(-10.5)[1 + (10.5) + (10.5)^2/2 + (10.5)^3/6 + (10.5)^4/24] = 0.978906\dots$ which is not bad...
- (d) Let Y denote the number of passengers who show up for the flight, regardless of where they came from. When the connecting flight is on time, 15 passengers are guaranteed to show up. Of the other 90, the number who show up is a binomial random variable Z with parameters $(90, 0.9)$. Thus, $Y = 15 + Z$ when the connecting flight is on time. Otherwise, when the connecting flight is late, only Z passengers show up, and hence $Y = Z$ in this case.
Thus, $P\{Y \leq 100 \mid \text{on time}\} = P\{Z \leq 85\} = 1 - P\{Z > 85\} = 0.95345\dots$ and $P\{Y \leq 100 \mid \text{late}\} = 1$.

9. Let A and B denote respectively the events that your **first** choice and your **final** choice is the curtain concealing the prize. $P(A) = 1/3$, $P(A^c) = 2/3$.
- (a) If you always switch, $P(B|A) = 0$, while $P(B|A^c) = 1$. Hence, $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 2/3$.
- (b) If you never switch, then $P(B|A) = 1$, while $P(B|A^c) = 0$. Hence, $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 1/3$.
- (c) If you decide at random, then $P(B|A) = P(B|A^c) = 1/2$ and $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 1/2$ also. Monty is correct in his assertion. (Would he lie to you? Besides, you saw it on **TV**, so it must be true!!!!)
10. This game is different from the one in Problem 6 in that *you have no idea what the rules of the game are*. If the man is intent on separating you from your money as quickly as possible, he will not offer the chance to switch unless you picked the shell hiding the pea in the first place! That is, if you picked the wrong shell, the man will reveal the pea and you will lose your bet. Of course, if you look like a person willing to play several rounds, the man may set you up by playing by Monty's rules (and allowing you to win with probability $2/3$) for some time. Then you will place a large bet on the wrong shell, and all of a sudden, you will not be given the choice of changing your bet! Personally, I would stick with the shell originally chosen since it gives me at least a $1/3$ probability of winning regardless of the man's strategy; your experience (and monetary losses) may vary.