

1. E, F, and G are events defined on a sample space
- (a) Only E occurs: EF^cG^c . Note that the expression E means that E occurs without excluding F or G; they may or may not have occurred. (b) Both E and G but not F: EF^cG .
- (c) At least one of the three occur. $E \cup F \cup G$. The more cumbersome expression $E \cup F \cup G \cup EF \cup EG \cup FG \cup EFG$ is equivalent to the simpler one (Prove it!)
- (d) At least two occur. $EF \cup EG \cup FG$. (e) All three occur. EFG . (f) None occur. $E^cF^cG^c$
- (g) At most one occurs. $E^cF^c \cup E^cG^c \cup F^cG^c$ (Compare (d). The given expression is just the one describing the event that at least two of the events E^c , F^c , G^c occurred (and hence at most one of E, F, G occurred!))
- (h) At most two occur. $E^c \cup F^c \cup G^c$ (Compare (c). This is just the event that at least one of the events E^c , F^c , G^c occurred (i.e. at most two of E, F, G occurred). Alternatively, $(EFG)^c$ since all three do not occur.
- (i) Exactly two occur. $EFG^c \cup EF^cG \cup E^cFG$ or $(EF \cup EG \cup FG) - EFG$.
- (j) At most three occur. Ω

- 2.(a) The binomial theorem states that $(1+x)^n = \sum_{i=0}^n \binom{n}{i} x^i$ where $\binom{n}{i}$, the coefficient of x^i , is the number of

events that contain i outcomes. Putting $x = 1$ gives $2^n = \sum_{i=0}^n \binom{n}{i}$, i.e. the total number of events is 2^n .

Putting $x = -1$ and noting that $(-1)^i$ is +1 or -1 according as i is even or odd, we get that

$$0 = \sum_{i \text{ even}} \binom{n}{i} - \sum_{i \text{ odd}} \binom{n}{i} \text{ which in turn implies that } \sum_{i \text{ even}} \binom{n}{i} = \sum_{i \text{ odd}} \binom{n}{i} = 2^{n-1} \text{ (Why is each sum } 2^{n-1}?)$$

Alternatively, suppose that **n is odd** and consider an event A of odd cardinality k. Then, A^c has even cardinality $n-k$. Thus, corresponding to every event of odd cardinality, there is an event of even cardinality (and vice versa). It follows that there must be equal numbers of each type of event. Unfortunately, this proof does not work for even n, but the following always works!

Let x denote one of the outcomes in Ω , and consider the 2^{n-1} events (subsets of Ω) that do not include x. Thus, $\emptyset$ is included among these events but Ω is not. Suppose that i such events are of odd cardinality and j of even (where $i+j = 2^{n-1}$). The remaining 2^{n-1} events all contain x, and can be thought of as being formed by including x in each of the 2^{n-1} events previously considered, e.g. changing $\{a,b,c,d\}$ to $\{a,b,c,d,x\}$ and $\emptyset$ to $\{x\}$ and $\{a,b,c,d,\dots\}$ to $\{a,b,c,d,\dots,x\} = \Omega$. But, including x in an event changes the cardinality from odd to even and vice versa. Thus, i events that include x have even cardinality and j have odd cardinality. Hence, $i+j = 2^{n-1}$ events are of even cardinality and $i+j = 2^{n-1}$ events are of odd cardinality.

- (b) Pair up each event A with its complement A^c . As a special case, Ω is paired with $\emptyset$, and note that we thus have matched up the 2^n events into 2^{n-1} pairs. Now, consider the sum of the probabilities of all 2^n events. Each pair contributes a total of $P(A) + P(A^c) = 1$ to the sum. Since there are 2^{n-1} such pairs, the sum of all the probabilities is 2^{n-1} . Dividing by 2^n , we find that the “average probability” is $1/2$. Note that we have not needed the assumption that all the n outcomes are equally likely — any assignment of probabilities p_1 ,

$p_2, \dots, p_n$ to the n outcomes (where each $p_i \geq 0$ and $\sum_{i=1}^n p_i = 1$) gives the same result that the “average

probability” is $1/2$; because it is always true that $P(A) + P(A^c) = 1$.

A longer way of obtaining the same result (with equally likely outcomes) is to note that an event containing i outcomes has probability i/n so that we are finding the sum

$$\sum_{i=0}^n \binom{n}{i} \times \frac{i}{n} = \sum_{i=1}^n \binom{n}{i} \times \frac{i}{n} = \sum_{i=1}^n \frac{n \times (n-1) \times \dots \times (n-i+1)}{1 \times 2 \times \dots \times i} \times \frac{i}{n} =$$

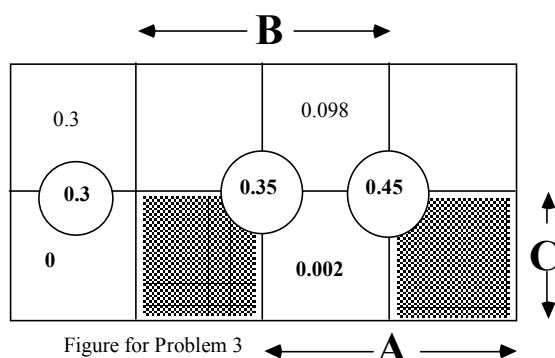
$$\sum_{i=1}^n \frac{(n-1) \times \dots \times (n-i+1)}{1 \times 2 \times \dots \times (i-1)} = \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}$$

Or, differentiate both sides in the statement of the binomial theorem and then set $x = 1$ to get $n(1+x)^{n-1} =$

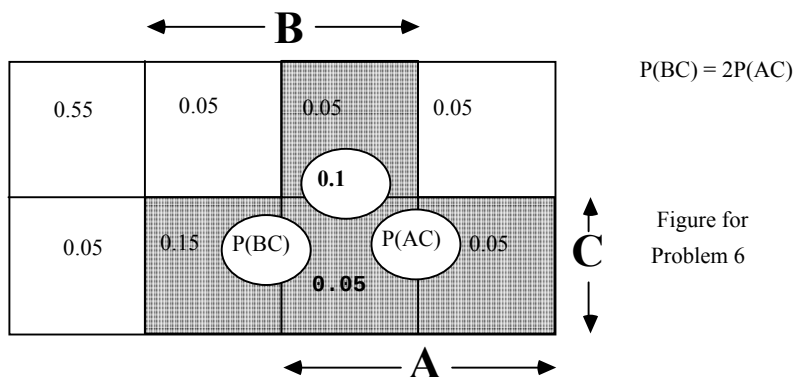
$$\sum_{i=1}^n \binom{n}{i} i x^{i-1} \Rightarrow n 2^{n-1} = \sum_{i=1}^n \binom{n}{i} \times i = \sum_{i=0}^n \binom{n}{i} \times i. \text{ This can also be interpreted to mean that adding up the}$$

cardinalities of all the subsets of Ω gives a total count of $|\Omega| \times 2^{|\Omega|-1}$ (There are $n \times 2^{n-1}$ ones in a truth-table!)

- (c) If n is even, then $\binom{n}{n/2}$ events have probability $1/2$. If n is odd, no event has probability $1/2$.
- (d) Let x denote the outcome that has occurred. Then, all the 2^{n-1} events that have occurred can be expressed as $\{x\} \cup A$ where A denotes a subset of the remaining $n-1$ outcomes. The sum of the cardinalities of all these 2^{n-1} sets is $2^{n-1} + (n-1) \times 2^{n-2}$ where the first term comes from the x in each set and the second from the 2^{n-1} subsets A . Divide by n to get the total probability and by 2^{n-1} to get that the average probability of the 2^{n-1} events that occurred is $(n+1)/(2n) = (1/2) + 1/(2n)$ which is just slightly larger than $1/2$.
- 3.(a) From DeMorgan's theorem and $P(A^c B^c) = 0.3$, we get that $P(A \cup B) = 0.7$.
 $P(AB) = P(A) + P(B) - P(A \cup B) = 0.1$. $P(AB^c) = P(A) - P(AB) = 0.35$.
- (b) $P(ABC^c) = P(AB) - P(ABC) = 0.098$. Note that C can be partitioned into ABC , $(A \oplus B)C$ and $A^c B^c C$, where $(A \oplus B)C$ is the shaded set in the figure shown. Since $P(A^c B^c C) = 0$, we get that $P(C) = P(ABC) + P(A \oplus B)C = 0.006$.
- (c) $P(A^c B^c C)$ and $P(A \cup C)$ cannot be computed.



- 4.(a) $A = \{d_1 d_2 \dots d_7 1 : d_i = \text{don't care if it is 0 or 1}\}$ giving $|A| = 2^7 = 128$; $P(A) = 2^7/2^8 = 1/2$.
- (b) The shift register contains 4 1's which can be chosen from the 8 positions in $\binom{8}{4} = \frac{8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4} = 70$ ways (the remaining 4 positions have 0's in them.) Hence, B contains 70 outcomes and $P(B) = 70/2^8 = 35/128$.
- (c) AB is the set of shift register contents with 4 1's, one of which is in the least insignificant bit position. The remaining 3 1's can be in any of $\binom{7}{3} = \frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35$ positions. Hence, $P(A \cap B) = 35/2^8$;
 $P(A \cup B) = P(A) + P(B) - P(A \cap B) = (128 + 70 - 35)/2^8 = 163/2^8$;
 $P(A \oplus B) = P(A) + P(B) - 2P(A \cap B) = (128 + 70 - 70)/2^8 = 1/2$.
- 5.(a) The outcomes are all the vectors of length 5 in which 3 entries are b and 2 are c . The days to serve broccoli can be chosen in $\binom{5}{3} = 10$ ways, so $|\Omega| = 10$.
- (b) Mom chooses broccoli with probability $3/5$ from the five choices she has on Monday. Alternatively, broccoli is served on two other days besides Monday, and these days can be chosen in $\binom{4}{2} = 6$ ways. Hence, $P(\text{broccoli on Monday}) = 6/10 = 3/5$.
- (c) The third broccolous day can be any of 3, and hence probability is $3/10$. (d) $1/10$.
6. We use the Venn diagram/Karnaugh map shown below.



$P(A \cup B) = P(A) + P(B) - P(AB) = 0.4$, and hence $P(A^c B^c) = 1 - P(A \cup B) = 0.6$
 $P(AB \cup AC \cup BC) = 0.3 = P(AB) + P(AC) + P(BC) - 2P(ABC) = 0.1 + 3P(AC) - 2 \times 0.05$. Hence, $P(AC) = 0.1$, $P(BC) = 0.2$, $P(AC \cup BC) = 0.1 + 0.2 - 0.05 = 0.25$. Thus, $P(A^c B^c C) = P(\text{snaps and crackles only}) = P(C) - P(AC \cup BC) = 0.05$, and $P(A^c B^c C^c) = P(\text{Rice Krispies}) = P(A^c B^c) - P(A^c B^c C) = 0.55$.
 $P(AB^c C^c) = P(A) - P(AB \cup AC) = 0.2 - 0.1 - 0.1 + 0.05 = 0.05$.

Similarly, $P(A^c B C^c) = P(B) - P(AB \cup BC) = 0.3 - 0.1 - 0.2 + 0.05 = 0.05$.

7.(a) $\Omega = \{C, H, A_1, T_1, T_2, A_2, N, O_1, O_2, G, A_3\}$. $P(\text{vowel}) = P(A_1, A_2, O_1, O_2, A_3) = 5/11$

(b) If $\Omega = \{A, C, G, H, N, O, T\}$, then $P(C) = P(H) = P(N) = P(G) = 1/11$, $P(T) = P(O) = 2/11$, and $P(A) = 3/11$.

(c) Now Ω has $\binom{11}{3} = 165$ outcomes in it with each outcome being a subset of size 3 from the Ω of part (a).

The number of such subsets that contain $\{C, A, T\}$ is $6 = 1 \times 3 \times 2$ since there is 1 choice for C, 3 for A, and 2 for T. Similarly, 6 subsets are $\{H, A, T\}$ and 6 are $\{T, A, N\}$ which also allows us to spell ANT.

Finally, the number of outcomes that are $\{O, A, T\}$ is $2 \times 3 \times 2 = 12$. Hence, the probability of being able to form one of the words CAT, HAT, OAT, TAN and ANT is $(6+6+6+12)/165 = 2/11$.

(d) Since letters are being replaced, there are $11 \times 11 \times 11$ outcomes in Ω . We can spell ANT or TAN if we happen to pick NAT, TAN, ANT, ATN, NTA, or TNA (in that order). $6 = 2 \times 3 \times 1$ outcomes in Ω give TAN (2 choices for T, 3 for A, 1 for N) and similarly for the other five orderings. The cases for CAT and HAT are similar. Similarly, $12 = 2 \times 3 \times 2$ outcomes give OAT and similarly for the other five orderings. Hence, $P(\text{subset spells CAT, HAT, OAT, TAN or ANT}) = (6 \times 3 + 12) \times 6 / 11 \times 11 \times 11 = 180/1331$.

(e) For **sampling with replacement**, part (d) has much of the answer. We get CAT, HAT, TAN or ANT *in that order* with 6 outcomes each, and OAT in that order with 12 outcomes. Hence, $P(\text{getting CAT, HAT, OAT, TAN or ANT in order}) = (6+6+6+6+12)/11 \times 11 \times 11 = 36/1331$.

For **sampling without replacement**, there are $11 \times 10 \times 9$ outcomes in Ω . Of these, $2 \times 3 \times 1$ are CAT, and similarly for HAT, ANT, and TAN. Of these 990 outcomes, $2 \times 3 \times 2$ are OAT. Hence the probability that we spell CAT, HAT, OAT, TAN or ANT as we draw is $36/990 = 2/55$.

8.(a) Let $q = 1-p$. Then,

$$P(F) = p + q^4 p + q^8 p + \dots = p[1 + q^4 + q^8 + \dots] = \frac{p}{1 - q^4}$$

$$P(W) = qp + q^5 p + q^9 p + \dots = qp[1 + q^4 + q^8 + \dots] = \frac{pq}{1 - q^4} = q \cdot P(F)$$

$$P(Ba) = q^2 p + q^6 p + q^{10} p + \dots = q^2 p[1 + q^4 + q^8 + \dots] = \frac{q^2 p}{1 - q^4} = q^2 \cdot P(F)$$

$$P(Be) = q^3 p + q^7 p + q^{11} p + \dots = q^3 p[1 + q^4 + q^8 + \dots] = \frac{q^3 p}{1 - q^4} = q^3 \cdot P(F)$$

Since $q < 1$, $P(F) > P(W) > P(Ba) > P(Be)$. Also,

$$P(F) + P(W) + P(Ba) + P(Be) = P(F) \cdot [1 + q + q^2 + q^3] = P(F) \cdot \frac{q^4 - 1}{q - 1} = \frac{p}{1 - q^4} \cdot \frac{q^4 - 1}{q - 1} = 1.$$

(b) Let $x = \ln 2$. Then, $P(\Omega) = \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{2(n!)}$ $= 2^{-1} \sum_{n=0}^{\infty} \frac{x^n}{n!} = 2^{-1} \cdot \exp(x) = 1$.

$$P(\text{outcome is an even number}) = \sum_{n=0}^{\infty} \frac{(\ln 2)^{2n}}{2(2n!)} = 2^{-1} \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n!)} = 2^{-1} \cdot \cosh(x) = \frac{5}{8} \text{ since } x = \ln 2.$$

9.(a) The number of weeks that your investment doubles in value is a *binomial* random variable Y with parameters $(5, 1/2)$. Since the investment halves in value during the remaining $5-Y$ weeks, and each halving cancels one doubling, we have that $X = 32 \cdot 2^{Y-5}$. The possible values of X are 1, 4, 16, 64, 256, and 1024, corresponding to $Y = 0, 1, 2, 3, 4, 5$ respectively.

(b) $P\{X = 1\} = P\{Y = 0\} = 1/32$. $P\{X = 4\} = P\{Y = 1\} = 5/32$. $P\{X = 16\} = P\{Y = 2\} = 10/32$.
 $P\{X = 64\} = P\{Y = 3\} = 10/32$. $P\{X = 256\} = P\{Y = 4\} = 5/32$. $P\{X = 1024\} = P\{Y = 5\} = 1/32$.

(c) $E[X] = 1 \cdot (1/32) + 4 \cdot (5/32) + 16 \cdot (10/32) + 64 \cdot (10/32) + 256 \cdot (5/32) + 1024 \cdot (1/32) = 97.65625$.

The TV commercial *understates* the performance — undoubtedly a first!

(d) $P\{X < 32\} = P\{X = 1\} + P\{X = 4\} + P\{X = 16\} = 1/2$.

(e) There is a 50% chance of losing money on this investment. Most people do not mind making investments if the amount at risk is small but the payoff from a win is enormous, e.g. lottery tickets are a losing bet for most buyers, but people don't mind an almost sure loss of a dollar because all they see is the huge payoff. Matters are considerably different when the amounts at risk are large, and most people tend to choose more conservatively. This explains why the local 7-11 sells lottery tickets but Neiman-Marcus does not.

10.(a) X takes on values $-6, 6, 12, 18$.

(b) If \$6 is bet on i , then you lose it if all three dice show one of the 5 non- i numbers.

Hence $P\{X = -6\} = 5^3/6^3 = \frac{125}{216}$. On the other hand, you win \$6 if one of the three dice shows i and the

other two have non- i numbers. Hence, $P\{X = 6\} = 3 \cdot (1 \cdot 5^2)/6^3 = \frac{75}{216}$. By a similar argument, $P\{X = 12\}$

is $3 \cdot (1^2 \cdot 5)/6^3 = \frac{15}{216}$, and $P\{X = 18\} = 1^3/6^3 = \frac{1}{216}$. Sanity check: $125 + 75 + 15 + 1 = 216$, so we have not left anything out.

(c) $E[X] = \sum u \cdot p(u) = \frac{125 \cdot (-6) + 75 \cdot 6 + 15 \cdot 12 + 1 \cdot 18}{216} = -\frac{102}{216} = -\frac{17}{36}$ i.e. a loss of roughly 47¢ per game.

(d) The chance of the three dice showing three different numbers is $\frac{6 \cdot 5 \cdot 4}{216} = \frac{5}{9}$. In this case, you come out even since you win \$3 on the three numbers showing, but lose \$3 on the three no-shows. The chance that the three dice show the same number is $6 \cdot \frac{1}{216} = \frac{1}{36}$ in which case you win \$3 on the winning number but lose \$5 on the 5 no-shows for a net loss of \$2. The probability that exactly two numbers are identical is thus $\frac{15}{36} = \frac{5}{12}$ in which case, you win \$2 on the pair and \$1 on the singleton, but lose \$4 on the other no-shows for a net loss of -1 . Thus, Y is a random variable taking on values 0, -1 , -2 , with probabilities as found above. (Remind me once again why you are bothering to play this game at all?), and its expected value is $E[Y] = \frac{0 \cdot 20 - 1 \cdot 15 - 2 \cdot 1}{36} = -\frac{17}{36}$ just as before. Splitting your bet six ways has no effect on your losses!