

- 1.(a) We use the principle of mathematical induction. Let $x \neq 1$. Then, the stated result is valid for $n = 1$ (since both sides are 1) and for $n = 2$ (since the left side is $1 + x$ and the right side $(1-x^2)/(1-x) = 1 + x$). So, let us assume that the result is true for $n = 1, 2, \dots, k$. Thus, $1 + x + x^2 + \dots + x^{k-1} = (1-x^k)/(1-x)$, and hence
- $$1 + x + x^2 + \dots + x^{k-1} + x^k = \frac{1-x^k}{1-x} + x^k = \frac{1-x^k + x^k - x^{k+1}}{1-x} = \frac{1-x^{k+1}}{1-x}.$$
- Therefore, the result holds for $n = k+1$ as well. Hence, by the principle of mathematical induction, the result holds for all $n \geq 1$. Alternatively, multiply the LHS by $(1-x) \neq 0$ and note that the resulting terms telescope to $1 - x^n$, that is, $(1 + x + x^2 + \dots + x^{n-1})(1-x) = 1 - x^n$, and the result follows on dividing both sides by $1-x$.

- (b) Clearly, if $|x| < 1$, $1 + x + x^2 + \dots$ converges to $\frac{1}{1-x}$. This series will be used repeatedly in the course.

- (c) The Taylor/MacLaurin series for $f(x)$ is $f(0) + \frac{x}{1!} \cdot f^{(1)}(0) + \frac{x^2}{2!} \cdot f^{(2)}(0) + \frac{x^3}{3!} \cdot f^{(3)}(0) + \dots$ where $f^{(i)}(0)$ denotes the i -th derivative of $f(x)$ evaluated at $x = 0$. Now, if $f(x) = (1+x)^n$, then it is easy to compute that $f^{(i)}(0) = n \cdot (n-1) \cdot \dots \cdot (n-i+1)$. Hence, we get

$$f(x) = (1+x)^n = 1 + \frac{n}{1!} \cdot x + \frac{n(n-1)}{2!} \cdot x^2 + \frac{n(n-1)(n-2)}{3!} \cdot x^3 + \dots$$

where the four terms shown are the 3rd degree Taylor polynomial for $(1+x)^n$.

- (d) If $g(x) = (1-x)^n$, then it is easy to compute that $g^{(i)}(0) = n \cdot (n-1) \cdot \dots \cdot (n-i+1)(-1)^i$. Hence, we get

$$g(x) = (1-x)^n = 1 - \frac{n}{1!} \cdot x + \frac{n(n-1)}{2!} \cdot x^2 - \frac{n(n-1)(n-2)}{3!} \cdot x^3 + \dots$$

- (e) True. $g^{(i)}(0) = (-1)^i \cdot f^{(i)}(0)$ and hence $f^{(i)}(0) + g^{(i)}(0)$ equals $2f^{(i)}(0)$ or 0 according as i is even or odd.

- 2.(a) My cheap calculator gives 0 as the answer in all cases. This seems reasonable because $\sin x \approx x$ for small x , and thus the function should have value nearly 0 as x approaches 0. However, the answers that you get *will* depend on the number of digits of precision used by your calculating device as well as the algorithm it uses to compute $\sin(x)$. For example, at $x = 10^{-7}$, the Unix high-precision calculator utility *bc* gives the following values: $\approx 10^9$ using 20 digits; $\approx 10^3$ using 25 digits, ≈ 0.34 using 30 digits and ≈ 0.33333333333334 using 45 or more digits. Thus, it would appear that the limit might be $1/3$. For the formal proof, note $\sin x = x - x^3/3! + x^5/5! - \dots \approx x - x^3/6$ for small x . Hence, for small x ,

$$\frac{1}{[\sin x]^2} - \frac{1}{x^2} \approx \frac{1}{x^2} \left[\frac{1}{(1 - x^2/6)^2} - 1 \right] = \frac{1}{x^2} [1 + 2x^2/6 + 3(x^2/6)^2 + \dots - 1] = \frac{1}{3} + \frac{3x^2}{36} \dots$$

Here we have used the series: $(1-y)^{-2} = 1 + 2y + 3y^2 + \dots$ which is valid for $|y| < 1$.

Remember this series; it and the series $(1-y)^{-1} = 1 + y + y^2 + y^3 + \dots$ will be used many times in the course.

Exercise: What do you get when you differentiate both sides of $(1-y)^{-1} = 1 + y + y^2 + y^3 + \dots$

$$\text{Thus, } \frac{1}{[\sin x]^2} - \frac{1}{x^2} \approx 1/3 + 3x^2/36 + \dots = 1/3 \text{ at } x = 0, \text{ i.e., } \lim_{x \rightarrow 0} \frac{1}{[\sin x]^2} - \frac{1}{x^2} = \frac{1}{3}$$

Those who prefer a purer approach should note that $\frac{1}{[\sin x]^2} - \frac{1}{x^2} = \frac{x^2 - \sin^2 x}{x^2 \sin^2 x}$ is of the form $0/0$ at $x = 0$.

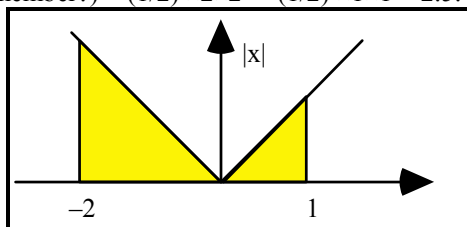
Thus, by (repeated application of) L'Hôpital's rule and the identity $\frac{d}{dx} \sin^2 x = \sin(2x)$ (huh??), we have

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1}{[\sin x]^2} - \frac{1}{x^2} &= \lim_{x \rightarrow 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \lim_{x \rightarrow 0} \frac{2x - \sin(2x)}{2x \sin^2 x + x^2 \sin(2x)} \\ &= \lim_{x \rightarrow 0} \frac{2 - 2\cos(2x)}{2\sin^2 x + 4x \sin(2x) + 2x^2 \cos(2x)} = \lim_{x \rightarrow 0} \frac{4\sin(2x)}{2\sin(2x) + 12x \cos(2x) - 4x^2 \sin(2x) + 4\sin(2x)} \\ &= \lim_{x \rightarrow 0} \frac{8\cos(2x)}{4\cos(2x) + 12\cos(2x) + 8\cos(2x) + x(\text{some other stuff})} = \frac{8}{24} = \frac{1}{3} \text{ just as before.} \end{aligned}$$

- (b) The function $f(x) = x^4(1.001)^{-x^2}$ can be written in the form $x^n \exp(-ax^2)$, where $a = \ln(1.001)$, $n = 4$. The function $x^n \exp(-ax^2)$ has local extrema at the points $x = -\infty, -(n/2a)^{1/2}, 0, (n/2a)^{1/2}, \infty$. In this case $n = 4$ and therefore maxima occur at $x = -44.7325, 44.7325$ (as can be verified by differentiating once again and checking the sign of the result at two points); the points $x = -\infty, 0, \infty$ represent minima. The maximum value (at both maxima) is 5.4188×10^5 .

- 3.(a) From the picture: $\int_{-2}^1 |x| dx =$ the shaded area = sum of the areas of two triangles

(area = $(1/2) \times \text{base} \times \text{altitude}$, remember?) = $(1/2) \times 2 \times 2 + (1/2) \times 1 \times 1 = 2.5$.



Those who insist on integration should note that $|x| = x$ if $x > 0$ and $|x| = -x$ if $x < 0$, and hence $\int_{-2}^1 |x| dx =$

$$\int_{-2}^0 -x dx + \int_0^1 x dx = \left. \frac{-x^2}{2} \right|_{-2}^0 + \left. \frac{x^2}{2} \right|_0^1 = 2.5. \text{ The answer is NOT } \int_{-2}^1 |x| dx = \left. \frac{|x|^2}{2} \right|_{-2}^1 = \frac{1^2}{2} - \frac{(-2)^2}{2} = -1.5$$

It is very important to learn the habit of drawing a sketch before tackling integrals in this course. It will save you from a lot of errors, and in some cases, it will ease your task considerably. The above is an example of this.

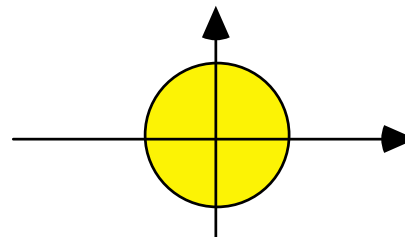
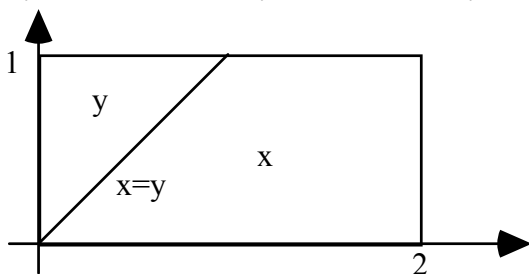
- (b) Put $y = 1 - x$. Then, $\int_0^1 x(1-x)^{19} dx = \int_1^0 (1-y)y^{19}(-dy) = \int_0^1 y^{19} - y^{20} dy = \frac{1^{20}}{20} - \frac{1^{21}}{21} = \frac{1}{420}$

- (c) Integrate by parts. $\int_0^1 x \exp(-x) dx = -x \exp(-x) \Big|_0^1 + \int_0^1 \exp(-x) dx = 1 - \frac{2}{e}$

4. (i) True: the chain rule gives $-g(-x)$ as the derivative. (ii) False: the chain rule gives $2xg(x^2)$ as the derivative. (iii) True. cf. (i) (iv) False: the antiderivative of $g(x^2)$ need not be related to $f(x^2)$ at all!

- 5.(a) $f(x,y)$ has values x and y in the regions shown in the left-hand figure below. Hence, we break up the double integral into two double integrals over the two regions. **Be sure that you understand how and why the limits get set up the way they have been.** Then, we compute the inner integrals in the two double integrals, and recombine the resulting single integrals into one, thus getting

$$\int_{y=0}^1 \int_{x=0}^y f(x,y) dx dy = \int_{y=0}^1 \int_{x=0}^y y dx dy + \int_{y=0}^1 \int_{x=y}^2 x dx dy = \int_{y=0}^1 y^2 + 2 - y^2/2 dy = 2\frac{1}{6}.$$



- (b) The region to be integrated over is the exterior of the circle of radius $\sqrt{2}$ centered at the origin. Thus, changing to polar coordinates, we get $\int \int (x^2 + y^2)^{-2} dx dy = \int_{\sqrt{2}}^{\infty} \int_0^{2\pi} r^{-4} r dr d\theta = 2\pi \int_{\sqrt{2}}^{\infty} r^{-3} dr = \pi/2$.