Agenda

Final Review I (Post-midterm2)

- Joint Probability
- Estimators
- Functions of RV
- Joint PDF of functions of RV

Joint PMF

If X and Y are discrete, joint PMF $p_{X,Y}(u,v) = P\{X = u, Y = v\}$
Compute via counting or series.

Marginalization

- Getting single RV PMF/ PDF from joint PMF/ PDF
- $p_X(u) = \sum_{v_j} p_{X,Y}(u, v_j)$ called “marginal PMF”

Conditional PMF

- $p_{Y|X}(v|u_0) = \frac{p_{X,Y}(u_0,v)}{p_X(u_0)}$

Example

Roll two fair dice, denote the output as Z_1 and Z_2
Let $X = |Z_1 - Z_2|$ and $Y = \min(Z_1, Z_2)$

- $p_{XY}(u, v)$

$$\begin{cases} \frac{1}{36}. & u=0 \quad v \in [1,6] \\ \frac{1}{18}, & u \in [1, 6-v], \quad v \in [1,5] \\ 0 & \text{otherwise} \end{cases}$$

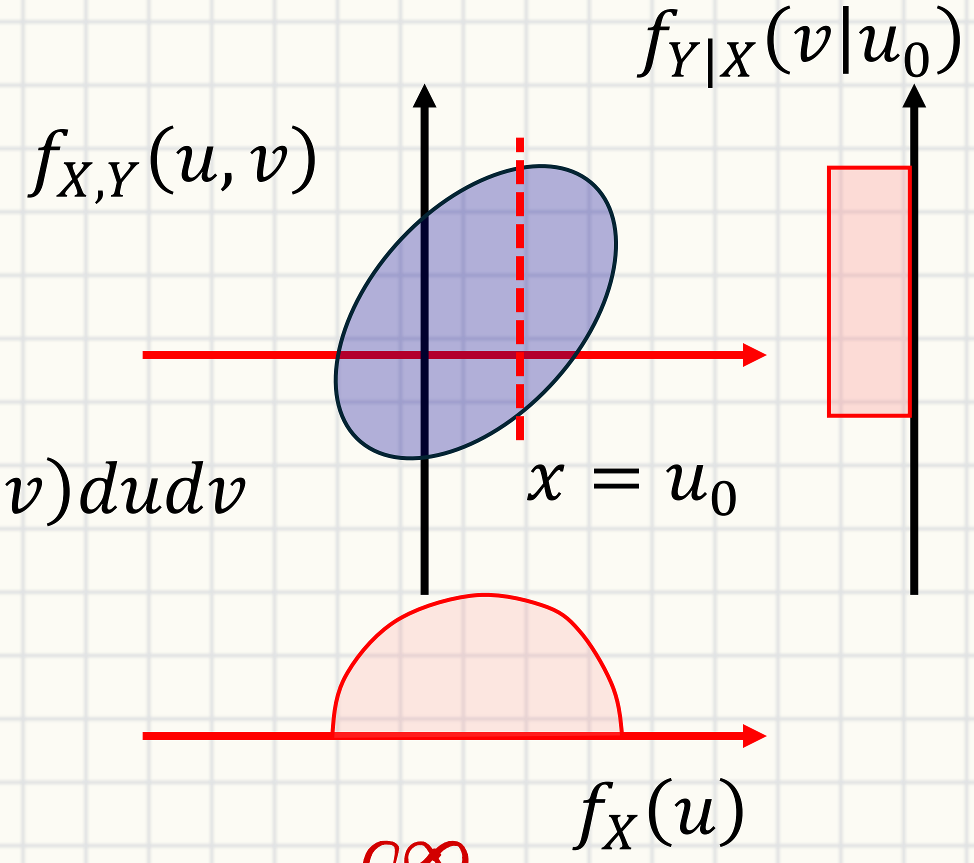
- $P\{X = Y\}$
- $P\{X > Y\}$
- $p_{Y|X}(v|2)$

$Y = \min(Z_1, Z_2)$

$X = |Z_1 - Z_2|$

X \ Y	1	2	3	4	5	6
0	(1,1)	(2,2)	(3,3)	- -	-	(6,6)
1	(1,2) (2,1)	(2,3) (3,2)	(3,4) (4,3)		(5,6) (6,5)	$\emptyset$?
2	(1,3) (3,1)					
3	(1,4) (4,1)					
4						
5	(1,6) (6,1)					

Joint Probability Density



X and Y are described by $f_{X,Y}(u,v)$

- $E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v) f_{X,Y}(u,v)\,du\,dv$
    - Double integrate over x and y

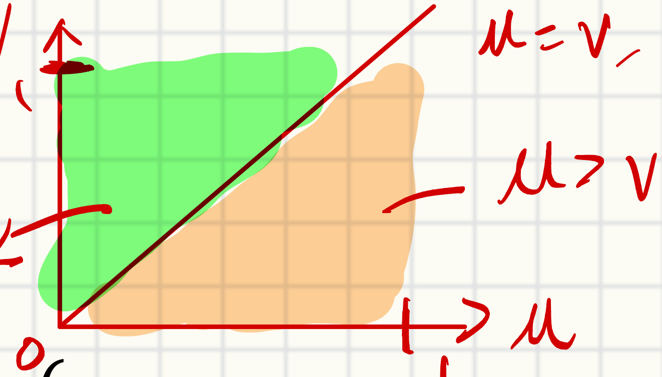
- $f_{Y|X}(v|u_0) = \dfrac{f_{X,Y}(u_0,v)}{f_X(u_0)}$
    - Conditional distribution
    - Useful in **unconstrained estimator**

$E[Y|X] = \int_{-\infty}^{\infty} v\, f_{Y|X}(v|u)\,dv$

- $f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv$ and $E[X] = \int_{-\infty}^{\infty} u \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv\,du$
    - Marginalization

$= \int_{-\infty}^{\infty} u\, f_X(u)\,du$

Example



$u>v$: $A(1-(u-v)) = A(1-u+v)$

$A(1-v+u)$

$$f_{X,Y}(u,v) = \begin{cases} A(1-|u-v|) & if \quad 0<u<1, 0<v<1 \\ 0 & else \end{cases}$$

$= \int_0^1 \int_u^1 A(1-v+u)\,dv\,du$

- Solve A $\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(1-|u-v|)\,du\,dv = \int_0^1\int_0^v A(1-v+u)\,du\,dv$

$+ \int_0^1\int_0^u A(1-u+v)\,dv\,du$

$= 1$

$\Rightarrow A = \frac{3}{2}$

- Find $f_X(u)$ and $f_Y(v)$

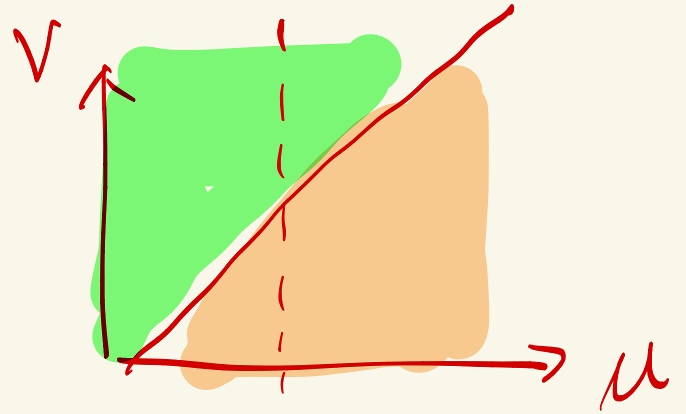
- Find $P\{X+Y<1|X>\frac{1}{2}\}$

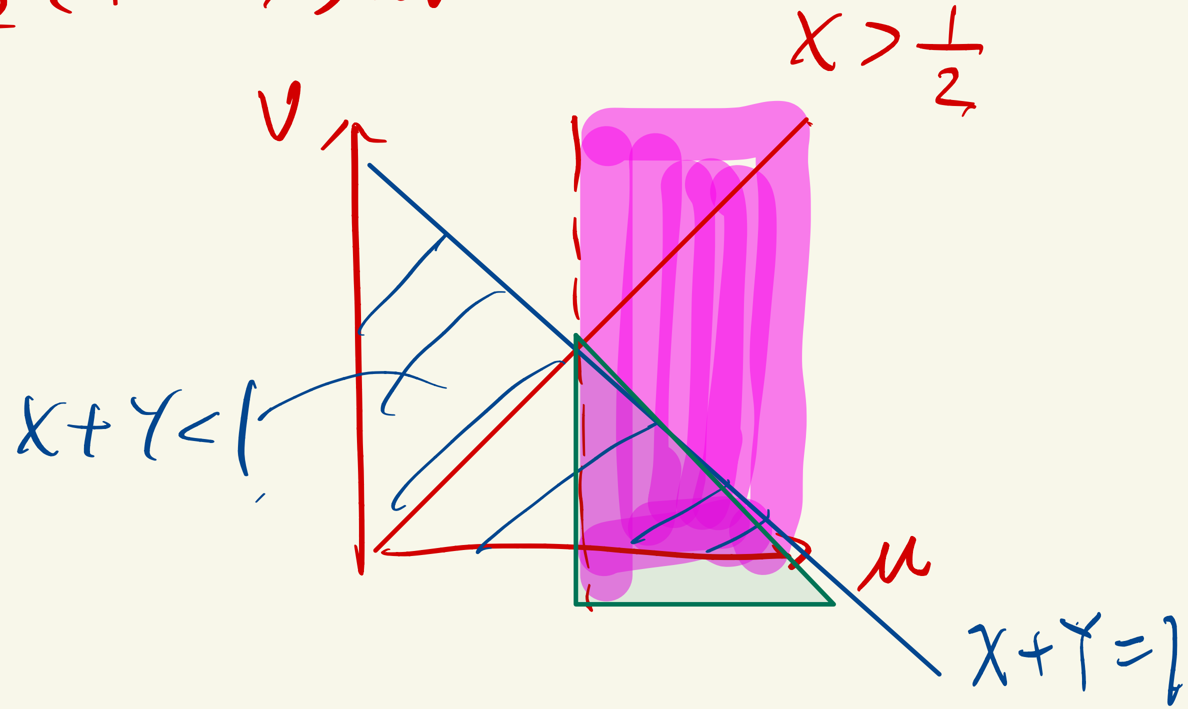
P

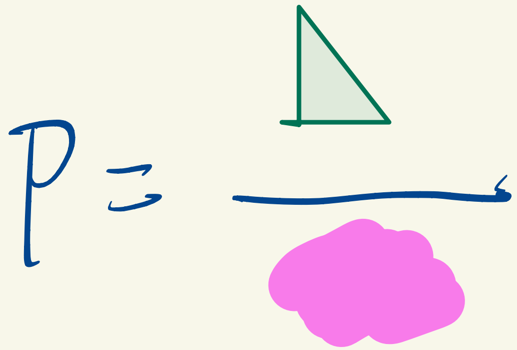
$$f_X(u) = \int_0^1 f_{XY}(u,v)\,dv$$

$$= \int_0^u \frac{3}{2}(1-u+v)\,dv + \int_u^1 \frac{3}{2}(1-v+u)\,dv$$

$$= \frac{3}{2}\left[-u^2+u+\frac{1}{2}\right]$$







Example

$$f_{X,Y}(u,v) = \begin{cases} c(1-y) & if \quad 0 \le x \le y \le 1 \\ 0 & else \end{cases}$$ $C = 6$

- Solve c $\quad \int_0^1 \int_0^v c(1-v) \, du\, dv = \int_0^1 \left[cu - cuv\right]_{u=0}^{v} dv$

$$= \int_0^1 cv - cv^2 \, dv$$

$$= \left[\frac{cv^2}{2} - \frac{cv^3}{3}\right]_{v=0}^{1}$$

$$= \frac{c}{2} - \frac{c}{3} = 1$$

- Find $f_X(u), f_Y(v), f_{X|Y}(u,v)$

- Find $E[X|Y = y]$

$$f_X(u) = \int_u^1 6(1-v)\,dv = \left[6v - 3v^2\right]_u^1$$

$$= (6-3) - (6u - 3u^2)$$

$$= 3 - 6u + 3u^2$$

$$f_Y(v) = \int_0^v 6(1-v)\,du$$

$$= \left[6u(1-v)\right]_{u=0}^{v} = 6v(1-v)$$

$$f_{X|Y}(u|v_0) = \frac{6(1-v_0)}{6v_0(1-v_0)} = \frac{1}{v_0} \qquad 0 \le u \le v_0$$

Estimators

- Constant estimator
  - $c^* = E[Y]$, MSE = $Var(Y)$
- Unconstraint estimator
  - $g^*(X) = E[Y|X]$ MSE=$E[Y^2] - E[(E[Y|X])^2] = E[Y^2] - E[\left(g^*(X)\right)^2]$
  - Best estimator, but requires $f_{Y|X}$
- Linear estimator
  - $\hat{E}[Y|X] = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(X - \mu_X) = \mu_Y + \rho_{X,Y}\sigma_Y(\frac{X-\mu_X}{\sigma_X})$
  - MSE= $\sigma_Y^2 - \frac{\left(Cov(X,Y)\right)^2}{Var(X)} = \sigma_Y^2(1 - \rho_{X,Y}^2)$

Example

$$f_{X,Y}(u,v) = \begin{cases} b(u+v) & if \quad 0 < u < 1, 0 < v < 1 \\ 0 & else \end{cases}$$

- Find c^* for Y

$$\int_0^1 \int_0^1 b(u+v)\, du\, dv = 1$$

$$\Rightarrow b = 1.$$

- Find $E[Y|X]$

$$f_Y = \int_0^1 (u+v)\, du = \left[\frac{u^2}{2} + uv \right]_{u=0}^{1}$$

$$= \frac{1}{2} + v.$$

- Find $\hat{E}[Y|X]$

$$\mu_Y = \int_0^1 v\left(\frac{1}{2} + v\right) dv.$$

$$= \left[\frac{v^2}{4} + \frac{v^3}{3} \right]_{v=0}^{1} = \frac{7}{12}$$

$$f_{Y|X} = \frac{f_{XY}(u,v)}{f_X(u)} = \frac{u+v}{\frac{1}{2}+u} = \frac{2u+2v}{1+2u}$$

$$E[Y|X] = \int_0^1 v\, f_{Y|X}(v|u)\,dv = \int_0^1 \frac{2uv+2v^2}{1+2u}\,dv$$

$$E[Y|X=u]$$

$$= \left[\frac{uv^2}{1+2u} + \frac{2v^3}{3(1+2u)}\right]_{v=0}^{1} = g(u)$$

$$\mu_X,\ \mu_Y,\ Cov(X,Y) = E[XY] - \mu_X\mu_Y$$

$$\mu_X = \frac{7}{12} \qquad \mu_Y = \frac{7}{12} \qquad Var(X)$$

$$E[XY] = \int_0^1\int_0^1 uv(u+v)\,du\,dv$$

Functions of joint RVs

- $W = g(X, Y)$
  - Find $F_W(c)$
  - $f_W(c) = F'_W(c)$

$X \leq Y + c$

  - Let $W = X - Y$
  - $F_W(c) = P\{X - Y \leq c\} = \int_{-\infty}^{\infty} \int_{-\infty}^{v+c} f_{XY}(u, v)\, du\, dv$

  - $f_W(c) = F'_W(c) =$

$$\int_{-\infty}^{\infty} f_{XY}(v+c, v)\, dv$$

Example

$f_{XY}(u, v) = 4uv$ if $0 \leq u, v \leq 1$, 0 else
Let $S = X + Y$, find $f_S(c)$ support : $0 \le S \le 2$.

$$\Rightarrow f_S(c) = \int_{-\infty}^{\infty} f_{XY}(u, c-u)\, du.$$

$0 \le c \le 2$, $\quad 0 \le u \le 1 \quad 0 \le c-u \le 1 \qquad u \ge c-1$

$$\begin{cases} 0 \le c \le 1 & f_S(c) = \int_0^c 4u(c-u)\, du. \\ 1 < c \le 2 & f_S(c) = \int_{c-1}^1 4u(c-u)\, du. \end{cases}$$

Functions of joint RVs

- Single RV $Y = g(X)$
  1. Find support of X and Y
  2. Find $F_Y(c)$ from integrating $f_X(x)$ over $\{x: g(x) \leq c\}$
  3. Get $f_Y = F_Y{}'$

- Multiple RV $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
  - $f_{W,Z}(\alpha, \beta) = \frac{1}{|\det(A)|} f_{X,Y}(u, v) = \frac{1}{|\det(A)|} f_{X,Y}(A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix})$

Generalize to one-to-one mapping (not in exam)

Suppose $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$ where $\det(A) \neq 0$

- $f_{W,Z}(\alpha, \beta) = \dfrac{1}{|\det(A)|} f_{X,Y}(u, v)$

Suppose $\begin{pmatrix} W \\ Z \end{pmatrix} = g\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right) = \begin{pmatrix} g_1(x, y) \\ g_2(x, y) \end{pmatrix}$

- $f_{W,Z}(\alpha, \beta) = \dfrac{1}{|\mathcal{J}|} f_{X,Y}(u, v)$
- $\mathcal{J} = \begin{bmatrix} \dfrac{\partial g_1(u,v)}{\partial u} & \dfrac{\partial g_1(u,v)}{\partial v} \\ \dfrac{\partial g_2(u,v)}{\partial u} & \dfrac{\partial g_2(u,v)}{\partial v} \end{bmatrix}$

W W

Z Z

Example

$W = 2X - Y, Z = X + 2Y$. Express $f_{W,Z}(\alpha, \beta)$ in terms of $f_{X,Y}$

- $\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$
- $Det(A) =$
- For $(W, Z) = (\alpha, \beta)$, $(X, Y) =$
- $f_{W,Z}(\alpha, \beta) =$