Last lecture

Correlation and covariance (Ch 4.8)

- Examples
- Sample mean& variance, unbiased estimator (Ex. 4.8.7)

Minimum mean square error estimation (Ch 4.9)

- Constant estimators
- Unconstrained estimators
- Linear estimators

Agenda

Minimum mean square error estimation (Ch 4.9)

- Recap
  - Constant estimators
  - Unconstrained estimators
  - Linear estimators
- Examples

Joint Gaussian Distribution (Ch 4.11)

- Motivation
- Facts
- Examples

Estimators Recap

- Constant estimator
  - $c^* = E[Y]$, MSE = $Var(Y)$
- Unconstraint estimator
  - $g^*(X) = E[Y|X]$ MSE=$E[Y^2] - E[(E[Y|X])^2]$
  - Best estimator, but requires $f_{Y|X}$
- Linear estimator
  - $\hat{E}[Y|X] = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(X - \mu_X) = \mu_Y + \rho_{X,Y}\sigma_Y(\frac{X-\mu_X}{\sigma_X})$
  - MSE= $\sigma_Y^2 - \frac{\left(Cov(X,Y)\right)^2}{Var(X)} = \sigma_Y^2(1 - \rho_{X,Y}^2)$

Example

Let $X = Y + N$, $Y \sim Exp(\lambda)$ and $N \sim N(0, \sigma_N^2)$. Assume Y and N are independent

- Find $\hat{E}[Y|X]$

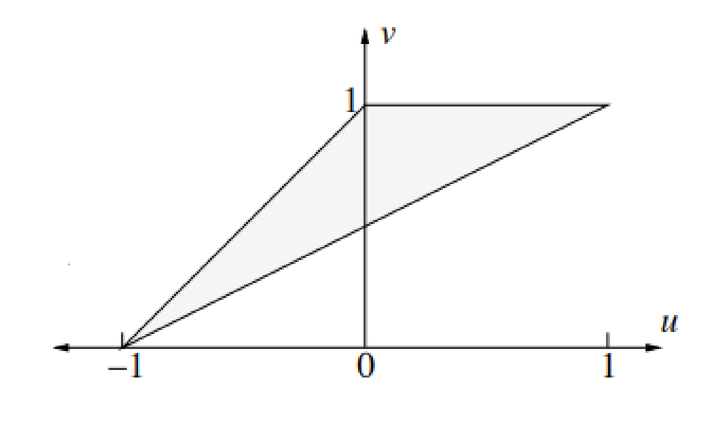
- Find the unconstrained estimator of Y

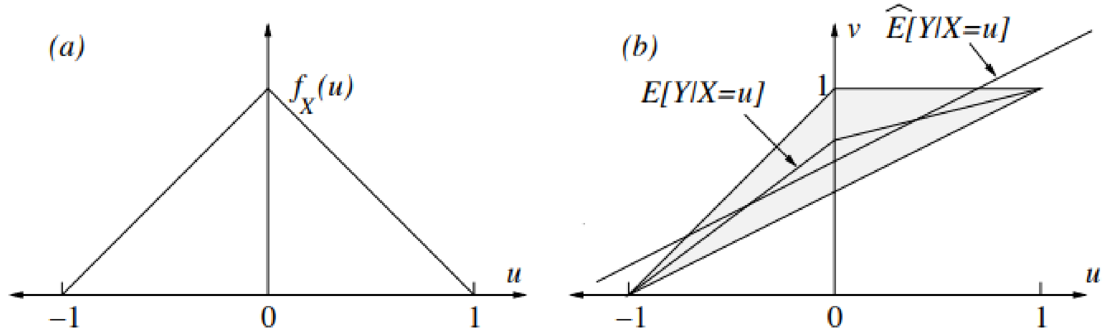
Example

Suppose X and Y are uniformly distributed in the triangle.

- Find $g^*(u) = E[Y|X = u]$ and the corresponding minimum MSE

- Find $\hat{E}[Y|X = u]$, and compute MSE





Joint Gaussian

Motivation

Describe the joint distribution of (Height, Weight)=(X, Y) of the class

- Metrics: $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho_{X,Y}$
- Is $aX + bY$ Gaussian?

Def. 4.11.1 RV X and Y are said to be **jointly Gaussian** if every linear combination $aX + bY$ is a Gaussian random variable

- $f_{X,Y}(u, v) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} exp(-\frac{\left(\frac{u-\mu_X}{\sigma_X}\right)^2+\left(\frac{v-\mu_Y}{\sigma_Y}\right)^2-2\rho\left(\frac{u-\mu_X}{\sigma_X}\right)\left(\frac{v-\mu_Y}{\sigma_Y}\right)}{2\sqrt{1-\rho^2}})$

Justifying a bivariate normal

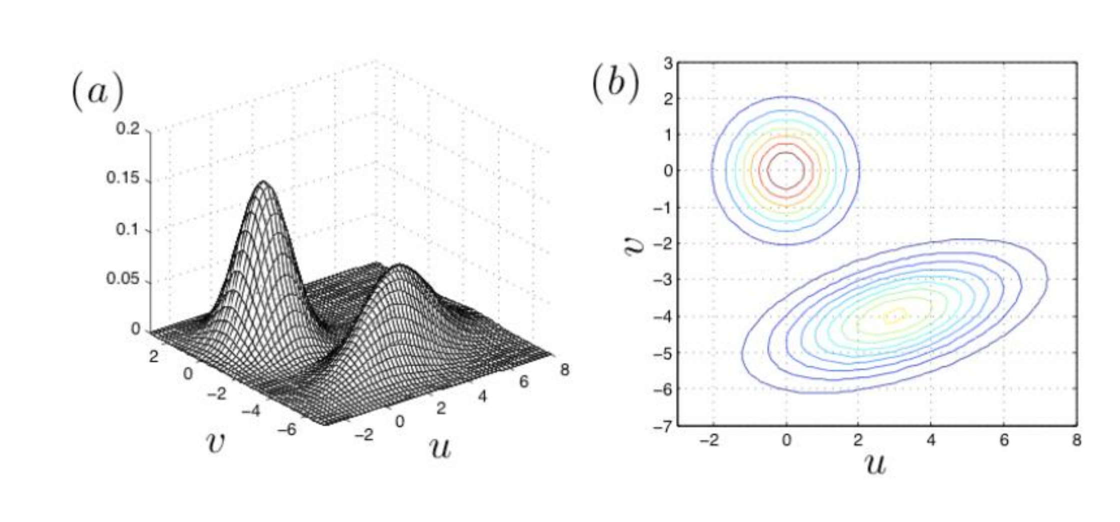
A bivariate normal $f_{X,Y}(u,v) = C \times exp\left(-P(u,v)\right)$

- $P(u,v) = au^2 + buv + cv^2 + du + ev + f$
- $P(u,v) \to \infty$ as $|u| + |v| \to \infty$
- $a, c > 0, b^2 - 4ac < 0$

Standard 2-d Normal to Bivariate Normal

Let W, Z be independent standard normal

- $f_{W,Z}(\alpha, \beta) = \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}} \frac{e^{-\frac{b^2}{2}}}{\sqrt{2\pi}} = \frac{e^{-\frac{\alpha^2+\beta^2}{2}}}{2\pi}$



- $\begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} W \\ Z \end{pmatrix} + \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$

- $A = \begin{bmatrix} \sqrt{\frac{\sigma_X^2(1+\rho)}{2}} & -\sqrt{\frac{\sigma_X^2(1-\rho)}{2}} \\ \sqrt{\frac{\sigma_Y^2(1+\rho)}{2}} & \sqrt{\frac{\sigma_Y^2(1-\rho)}{2}} \end{bmatrix}$

Facts

If X and Y forms the bivariate normal with $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho$

- $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$
- $aX + bY$ is a Gaussian for any $a, b \in \mathbb{R}$
- $\rho = \rho_{X,Y}$, X and Y are independent iif $\rho = 0$
- $E[Y|X] = \hat{E}[Y|X]$
- $(Y|X = u) \sim N(\hat{E}[Y|X = u], \sigma_e^2)$

Example

Let X and Y be jointly Gaussian with mean $(0,0)$. $\sigma_X^2 = 5$, $\sigma_Y^2 = 2$ and $Cov(X,Y) = -1$. Find $P\{X + 2Y \geq 1\}$ in terms of Φ function

- $Z = X + 2Y$ is Gaussian
- $\mu_Z =$
- $Var(Z) =$

Example

Let X and Y be jointly Gaussian RV with mean 0, variance 1, and $Cov(X, Y) = \rho$. Find $E[Y^2 | X]$

- $\hat{E}[Y|X] =$
- For any RV, $Var(Z) = E[Z^2] - (E[Z])^2$

Example

Let X and Y be jointly Gaussian RV with mean 0, variance 1, and $Cov(X, Y) = 0.5$. Find

- $Var(3X - 2Y)$
- $P\{(3X - 2Y)^2 \leq 28\}$ in terms of Φ
- $E[Y|X = 3]$

Quick questions

Which plot shape best represents the contour lines of a bivariate Gaussian?
A. Circles B. Ellipses C. Triangles D. Rectangles

If two Gaussian random variables X and Y are independent, what must be true?
A. $\rho_{XY} = 0$ B. Their means must be equal C. Their variances must be equal D. Nothing specific

Suppose (X,Y) is jointly Gaussian. If ρ=0, what is the resulting joint density shape?
A. A tilted ellipse B. A vertical ellipse C. A circle D. A horizontal straight line