Last lecture

Correlation and covariance (Ch 4.8)

- Examples
- Sample mean& variance, unbiased estimator (Ex. 4.8.7)

Minimum mean square error estimation (Ch 4.9) $E[Y|X]$

- Constant estimators $E[Y]$
- Unconstrained estimators $E[Y|X] = g^*(X)$
- Linear estimators

$\hat{E}[Y|X] = aX+b.$

Agenda

Minimum mean square error estimation (Ch 4.9)

- Recap
  - Constant estimators
  - Unconstrained estimators
  - Linear estimators
- Examples

Joint Gaussian Distribution (Ch 4.11)

- Motivation
- Facts
- Examples

Estimators Recap

- Constant estimator
  - $c^* = E[Y]$, MSE = $Var(Y)$
    $= E[Y^2] - (E[Y])^2$
- Unconstraint estimator
  - $g^*(X) = E[Y|X]$ MSE=$E[Y^2] - \underbrace{E[(E[Y|X])^2]}_{RV}$
  - Best estimator, but requires $f_{Y|X}$
- Linear estimator $ax+b$
  - $\hat{E}[Y|X] = \left(\mu_Y + \frac{Cov(X,Y)}{Var(X)}(X - \mu_X)\right) = \mu_Y + \rho_{X,Y}\sigma_Y(\frac{X-\mu_X}{\sigma_X})$
  - MSE= $\sigma_Y^2 - \frac{(Cov(X,Y))^2}{Var(X)} = \sigma_Y^2(1 - \rho_{X,Y}^2)$ $< \sigma_Y^2$ if $\rho_{X,Y} = 1/-1$ $\Rightarrow$ MSE $= 0$.

Example

Let $X = Y + N$, $Y \sim Exp(\lambda)$ and $N \sim N(0, \sigma_N^2)$. Assume Y and N are independent

- Find $\hat{E}[Y|X] = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(X - \mu_X)$

$$\begin{cases} \mu_Y = E[Y] = \frac{1}{\lambda} \\ \mu_X = E[Y+N] = \mu_Y + \mu_N = \frac{1}{\lambda} \end{cases}$$

$Cov(X,Y)$

$= Cov(Y+N, Y)$

$= Var(Y) + \underbrace{Cov(N,Y)}_{0}$

$= \frac{1}{\lambda^2}$

- Find the unconstrained estimator of Y

$$\mathrm{Var}(X) = \mathrm{Cov}(Y+N, Y+N) = \mathrm{Var}(Y) + \mathrm{Var}(N)$$

$$= \frac{1}{\lambda^2} + \sigma_N^2$$

$$\hat{E}[Y|X] = \frac{1}{\lambda} + \frac{\frac{1}{\lambda^2}}{\frac{1}{\lambda^2} + \sigma_N^2}\left(X - \frac{1}{\lambda}\right) = \frac{X + \lambda\sigma_N^2}{1 + \lambda^2\sigma_N^2}$$

$$E[Y|X] \Rightarrow Y > 0$$

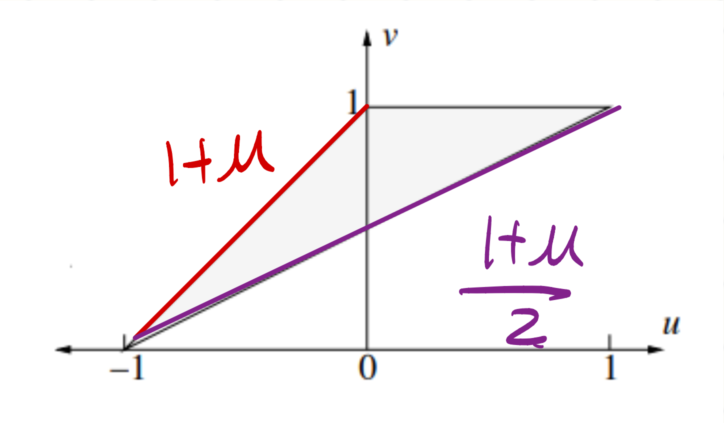
$$g^*(X) = \max(0, \hat{E}[Y|X])$$

$$E[(Y - g(X))^2] \leq E[(Y - \hat{E}[Y|X])^2]$$

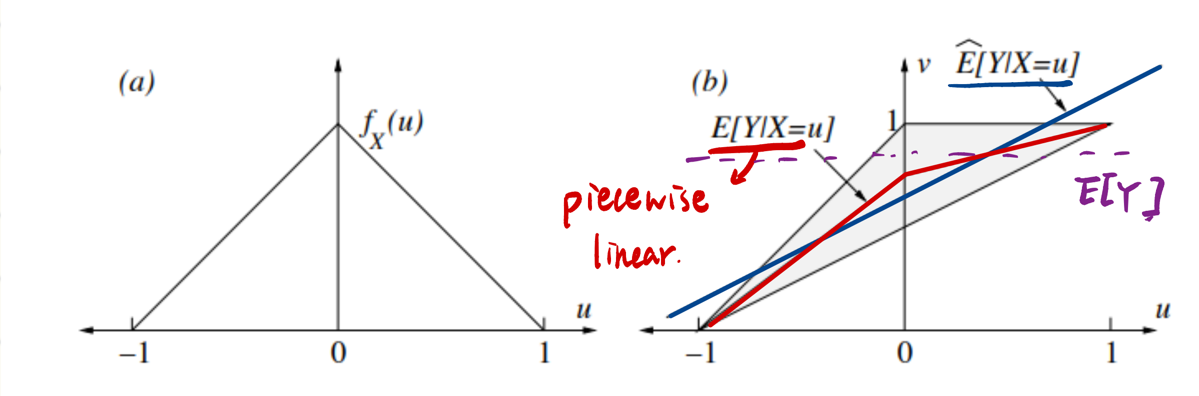
Example

Suppose X and Y are uniformly distributed in the triangle.

- Find $g^*(u) = E[Y|X = u]$ and the corresponding minimum MSE



- Find $\hat{E}[Y|X = u]$, and compute MSE



$$f_{Y|X}(v|u) \begin{cases} \text{Uniform}\left(\frac{1+u}{2},\ 1+u\right) & \text{if } -1<u<0 \\ \text{Uniform}\left(\frac{1+u}{2},\ 1\right) & \text{if } 0\le u<1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y|X] = g^*(u) = \int_{-\infty}^{\infty} v\, f_{Y|X}(v|u)\, dv$$

$$= \begin{cases} \frac{3+3u}{2} & \text{if } -1<u<0 \\ \frac{3+u}{2} & \text{if } 0\le u<1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\hat{E}[Y|X] = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(X - \mu_X)$$

$\mu_Y = \ldots$ $Cov(X,Y) = \ldots$

X Tested

Plan April 28 Bivariate / Large number

30 Expectation Maximization

May 5 mid 1/2 review

7 post mid 2 review

Joint Gaussian

Motivation

Describe the joint distribution of (Height, Weight)=(X, Y) of the class

- Metrics: $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho_{X,Y}$
- Is $aX + bY$ Gaussian?

$X \sim N(\mu_X, \sigma_X^2)$

$Y \sim N(\mu_Y, \sigma_Y^2)$

Def. 4.11.1 RV X and Y are said to be **jointly Gaussian** if every linear combination $aX + bY$ is a Gaussian random variable

- $f_{X,Y}(u,v) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} exp(-\frac{\left(\frac{u-\mu_X}{\sigma_X}\right)^2 + \left(\frac{v-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{u-\mu_X}{\sigma_X}\right)\left(\frac{v-\mu_Y}{\sigma_Y}\right)}{2\sqrt{1-\rho^2}})$

Justifying a bivariate normal

pdf of jointly Gaussian $X, Y \to aX+bY$

A bivariate normal $f_{X,Y}(u,v) = C \times exp(-P(u,v))$

- $P(u,v) = au^2 + buv + cv^2 + du + ev + f$
- $P(u,v) \to \infty$ as $|u| + |v| \to \infty$ $\Rightarrow \iint f_{XY}\, du\, dv = 1$
- $a, c > 0,\ b^2 - 4ac < 0$

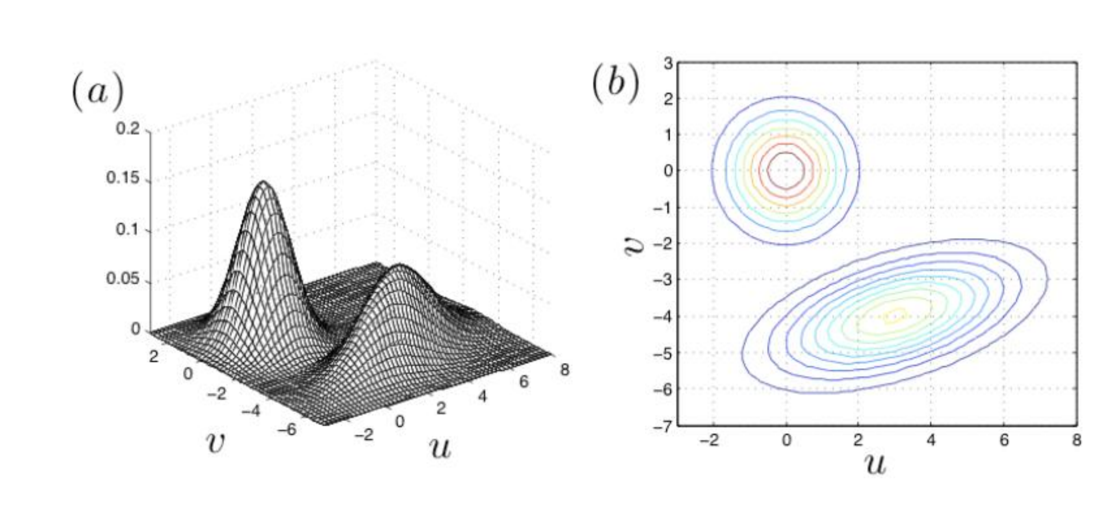
$-1 \le \rho \le 1$

Standard 2-d Normal to Bivariate Normal

Let W, Z be independent standard normal

- $f_{W,Z}(\alpha,\beta) = \dfrac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}}\dfrac{e^{-\frac{b^2}{2}}}{\sqrt{2\pi}} = \dfrac{e^{-\frac{\alpha^2+\beta^2}{2}}}{2\pi}$

$\Rightarrow$ W, Z are jointly Gaussian



- $\begin{pmatrix} X \\ Y \end{pmatrix} = A\begin{pmatrix} W \\ Z \end{pmatrix} + \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$

- $A = \begin{bmatrix} \overset{a_{11}}{\sqrt{\dfrac{\sigma_X^2(1+\rho)}{2}}} & \overset{a_{12}}{-\sqrt{\dfrac{\sigma_X^2(1-\rho)}{2}}} \\ \overset{a_{21}}{\sqrt{\dfrac{\sigma_Y^2(1+\rho)}{2}}} & \overset{a_{22}}{\sqrt{\dfrac{\sigma_Y^2(1-\rho)}{2}}} \end{bmatrix}$

$\Rightarrow$ $Var(X) = \sigma_X^2$

$Var(Y) = \sigma_Y^2$

$$\mathrm{Var}(X) = \mathrm{Cov}(a_{11}W + a_{12}Z,\ a_{11}W + a_{12}Z)$$

$$= a_{11}^2 \sigma_W^2 + a_{12}^2 \sigma_Z^2$$

$$= \frac{\sigma_X^2(1+\rho)}{2} + \frac{\sigma_X^2(1-\rho)}{2} = \sigma_X^2$$

$$\rho_{XY} = \frac{\mathrm{Cov}(\overset{X}{a_{11}W + a_{12}Z},\ \overset{Y}{a_{21}W + a_{22}Z})}{\sigma_X \sigma_Y} = \rho$$

Facts

If X and Y forms the bivariate normal with $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho$

- $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$

  $aX+bY$ $(a,b)=(1,0)$ or $(0,1)$

- $aX + bY$ is a Gaussian for any $a, b \in \mathbb{R}$
- $\rho = \rho_{X,Y}$, X and Y are independent iif $\rho = 0$

  $f_{XY}(u,v) = \underline{f_X(u)} f_{Y|X}(v|u)$

  $\sim N(\mu_X, \sigma_X^2)$

- $E[Y|X] = \hat{E}[Y|X]$
- $(Y|X = u) \sim N(\hat{E}[Y|X = u], \sigma_e^2)$

  $\sigma_Y^2(1-\rho^2)$

  $N(\mu_Y + \rho\, \mu.., \sigma_Y^2(1-\rho^2))$

Example

Let X and Y be jointly Gaussian with mean $(0,0)$. $\sigma_X^2 = 5$, $\sigma_Y^2 = 2$ and $Cov(X,Y) = -1$. Find $P\{X + 2Y \geq 1\}$ in terms of Φ function

- $Z = X + 2Y$ is Gaussian
- $\mu_Z = \mu_X + 2\mu_Y = 0$
- $Var(Z) =$

$Cov(X+2Y, X+2Y) = Var(X) + 4\,Cov(X,Y) + 4\,Var(Y)$

$= 5 + (-4) + 4 \times 2 = 9. \quad Z \sim N(0, 9)$

$P\left\{\frac{Z}{3} \geq \frac{1}{3}\right\} = Q\left(\frac{1}{3}\right) = 1 - \Phi\left(\frac{1}{3}\right)$

Example

$Y|X \sim N$

$Y^2|X \nsim N$

Let X and Y be jointly Gaussian RV with mean 0, variance 1, and $Cov(X,Y) = \rho$. Find $E[Y^2|X]$

- $\hat{E}[Y|X] = \cancel{\mu_Y}^{0} + \frac{\cancel{Cov(X,Y)}^{\rho}}{\cancel{Var(X)}^{1}}(X - \cancel{\mu_X}^{0}) = \rho X$
- For any RV, $Var(Z) = E[Z^2] - (E[Z])^2$

Let $Z = Y|X$

$$Var(Y|X) = E[Y^2|X] - (\underbrace{E[Y|X]}_{\rho X})^2$$

MSE of $\hat{E}[Y|X] = \sigma_Y^2(1-\rho^2)$

$$E[Y^2 | X] = \sigma_Y^2 (1-\rho^2) + \rho^2 X^2$$

Example

Let X and Y be jointly Gaussian RV with mean 0, variance 1, and $Cov(X, Y) = 0.5$. Find

- $Var(3X - 2Y)$

- $P\{(3X - 2Y)^2 \leq 28\}$ in terms of Φ

- $E[Y|X = 3]$ $= \hat{E}[Y|X=3] = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(3 - \mu_X)$

$= 1.5$

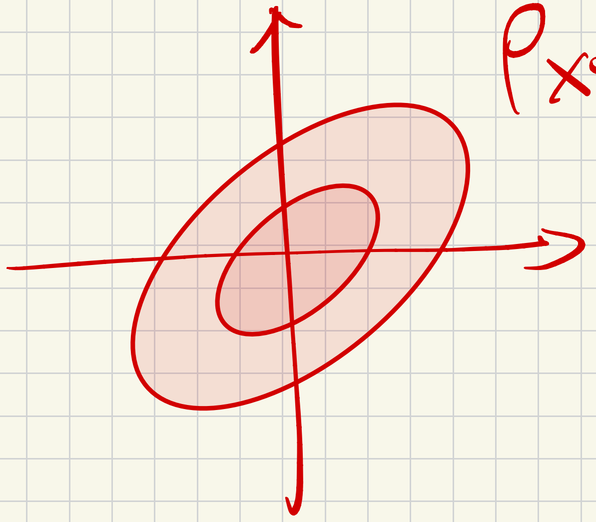
Quick questions

Which plot shape best represents the contour lines of a bivariate Gaussian?
A. Circles B. Ellipses C. Triangles D. Rectangles

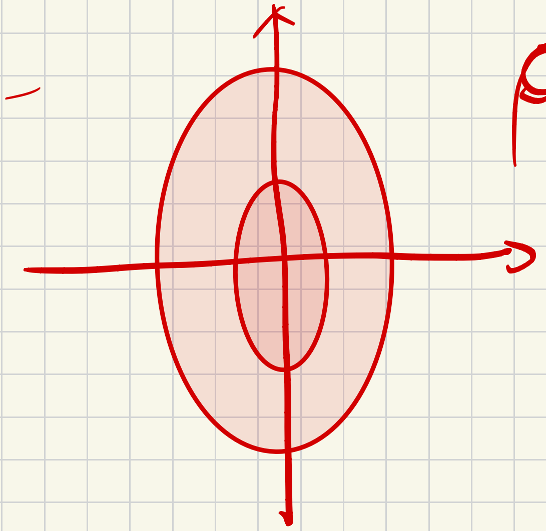
If two Gaussian random variables X and Y are independent, what must be true?
A. $\rho_{XY} = 0$ B. Their means must be equal C. Their variances must be equal D. Nothing specific

Suppose (X,Y) is jointly Gaussian. If ρ=0, what is the resulting joint density shape?
A. A tilted ellipse B. A vertical ellipse C. A circle D. A horizontal straight line

$\sigma_Y^2 = 0$



$\rho_{XY} > 0$



$\rho = 0$