Last lecture

Buffon's Problem

Joint pdfs of functions of RV (Ch 4.7)

- Linear mapping function (Ch 4.7.1)
- One to one/ Multiple to one functions (Ch 4.7.2-3)
  - Will not be tested

Correlation and covariance (Ch 4.8)

- Definition
- Properties
- Examples

Agenda

Correlation and covariance (Ch 4.8)

- Examples
- Sample mean& variance, unbiased estimator (Ex. 4.8.7)

Minimum mean square error estimation (Ch 4.9)

- Constant estimators
- Unconstrained estimators
- Linear estimators

Properties

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY] - \mu_X\mu_Y$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

Some properties for independent and uncorrelated

- $Cov(X+Y, U+V) = Cov(X,U) + Cov(X,V) + Cov(Y,U) + Cov(Y,V)$
- $Cov(aX+b, cY+d) = acCov(X,Y)$
- If X and Y are independent
  - $Var(X+Y) = Cov(X+Y, X+Y) = Cov(X,X) + Cov(Y,Y)$

Slido

Select those are uncorrelated

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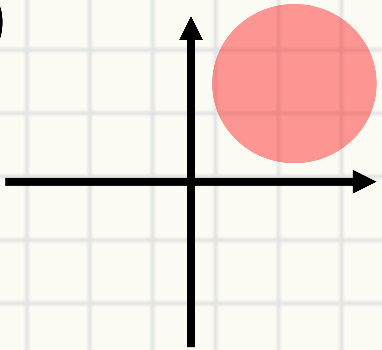

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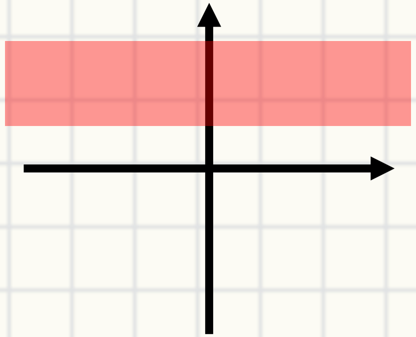

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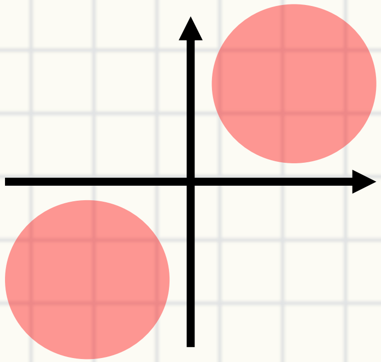

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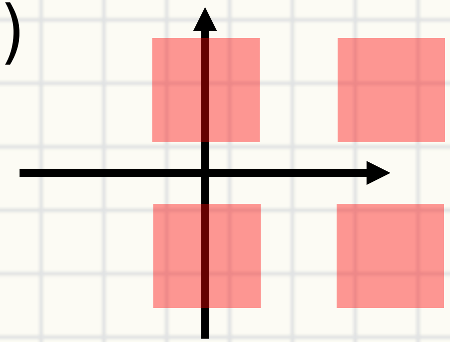

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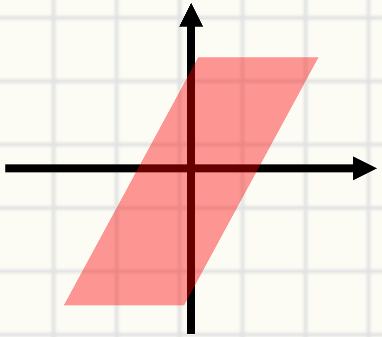

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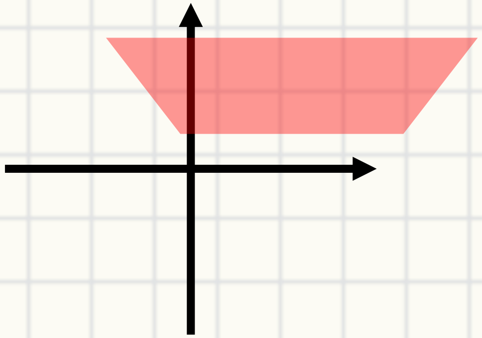


Example

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY] - \mu_X\mu_Y$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

Simplify the following expressions:

- $Cov(8X+3, 5Y-2)$
- $Cov(10X-5, -3X+15)$
- $Cov(X+2, 10X-3Y)$
- $\rho_{10X,Y+4}$

Example

Suppose the covariance matrix of RV vector $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ is $\begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

- Find $Cov(X_1 + X_2, X_1 + X_3)$
- Find a s.t. $X_2 - aX_1$ is uncorrelated with X_1
- Find ρ_{X_1, X_2}
- Find $Var(X_1 + X_2 + X_3)$

Sample Mean and Variance

Suppose $X_1 \ldots X_n$ are independent and identical distributed(i.i.d.) RVs with unknown mean μ and variance σ^2 (May not be Gaussian)

- Estimate μ and σ^2 by
- $\hat{X} =$
- $\widehat{\sigma^2} =$
- Unbiased: $E\left[\hat{Y}\right] = Y$
- Is sample mean and sample variance unbiased?

Minimum Mean Square Error Estimation

Constant Estimator

Given RV Y, if we know f_Y

- Constant estimator: constant $\delta^* = argmin_\delta E[(Y-\delta)^2]$
- $E[(Y-\delta)^2] = E[Y^2] - 2\delta E[Y] + \delta^2$
- Taking derivative w.r.t δ
- $\delta^* =$
- MSE $E[(Y-\delta^*)^2] =$

Unconstrained Estimator

Given RV X, Y, if we know $f_{X,Y}$ and some observations X

- Unconstrained Estimator - $g^*(X) = argmin_g\ E\left[\left(Y - g(X)\right)^2\right]$
- For example, say $X = 10$
  - $g^*(X) = E[Y|X = 10] = \int_{-\infty}^{\infty} v f_{Y|X}(v|u = 10)dv$
  - MSE = $E[(Y - E[Y|X = u])^2|X = u] =$
  - General form
  - MSE $E[(Y - E[Y|X])^2] =$

Linear Estimator

What if we do not know $f_{X,Y}$ or $f_{Y|X}$ is hard to compute?

- Let $g^*(X) = aX + b$, find $argmin_{(a,b)}\, E[\left(Y - (aX + b)\right)^2]$
- Can also be written as $argmin_{(a,b)}\, E[\left((Y - aX) - b\right)^2]$
  - b is the constant estimator of $Y - aX$
  - $b =$

Linear Estimator

$b = \mu_Y - a\mu_X$

What if we do not know $f_{X,Y}$ or $f_{Y|X}$ is hard to compute?

- Let $g^*(X) = aX + b$, find $argmin_{(a,b)}\, E[\left(Y - (aX + b)\right)^2]$
- MSE $E\left[\left(Y - \mu_Y - a(X - \mu_X)\right)^2\right] = Var(Y - aX) =$
- $\hat{E}[Y|X] = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(X - \mu_X) = \mu_Y + \rho_{X,Y}\sigma_Y(\frac{X-\mu_X}{\sigma_X})$
- MSE = $\sigma_Y^2 - \frac{\left(Cov(X,Y)\right)^2}{Var(X)} = \sigma_Y^2(1 - \rho_{X,Y}^2)$

Estimators Recap

- Constant estimator
  - $c^* = E[Y]$, MSE = $Var(Y)$

- Unconstraint estimator
  - $g^*(X) = E[Y|X]$ MSE=$E[Y^2] - E[(E[Y|X])^2]$
  - Best estimator, but requires $f_{Y|X}$

- Linear estimator
  - $\hat{E}[Y|X] = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(X - \mu_X) = \mu_Y + \rho_{X,Y}\sigma_Y(\frac{X-\mu_X}{\sigma_X})$
  - MSE$= \sigma_Y^2 - \frac{\left(Cov(X,Y)\right)^2}{Var(X)} = \sigma_Y^2(1 - \rho_{X,Y}^2)$

Example

Let $X = Y + N$, $Y \sim Exp(\lambda)$ and $N \sim N(0, \sigma_N^2)$. Assume Y and N are independent

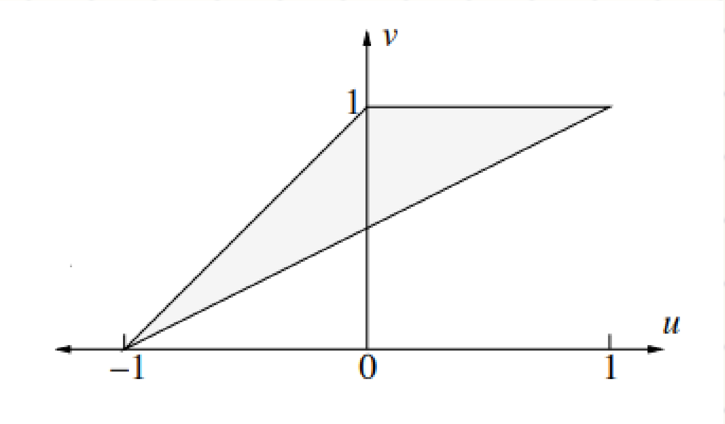
- Find $\hat{E}[Y|X]$

- Find the unconstrained estimator of Y

Example

Suppose X and Y are uniformly distributed in the triangle.

- Find $g^*(u) = E[Y|X = u]$ and the corresponding minimum MSE



- Find $\hat{E}[Y|X = u]$, and compute MSE