Last lecture

Buffon's Problem

Joint pdfs of functions of RV (Ch 4.7)

- Linear mapping function (Ch 4.7.1)
- One to one/ Multiple to one functions (Ch 4.7.2-3)
    - Will not be tested

Correlation and covariance (Ch 4.8)

- Definition
- Properties
- Examples

Agenda

Correlation and covariance (Ch 4.8)

- Examples
- Sample mean& variance, unbiased estimator (Ex. 4.8.7)

Minimum mean square error estimation (Ch 4.9)

- Constant estimators
- Unconstrained estimators
- Linear estimators

Properties

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY] - \mu_X\mu_Y$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

Some properties for independent and uncorrelated

- $Cov(X+Y, U+V) = Cov(X,U) + Cov(X,V) + Cov(Y,U) + Cov(Y,V)$

- $Cov(aX+b, cY+d) = acCov(X,Y)$

- If X and Y are independent
  - $Var(X+Y) = Cov(X+Y, X+Y) = Cov(X,X) + Cov(Y,Y)$

$= Var(X) + Var(Y)$

Slido

$E[XY]-\mu_X\mu_Y=0$.

sufficient. $\forall x$ if $E[X(Y-\mu_Y)]=0$

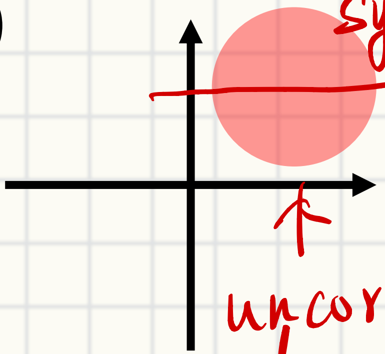
$\Rightarrow p\{x,\mu_Y+k\}=p\{x,\mu_Y-k\}$

Select those are uncorrelated

#4212882

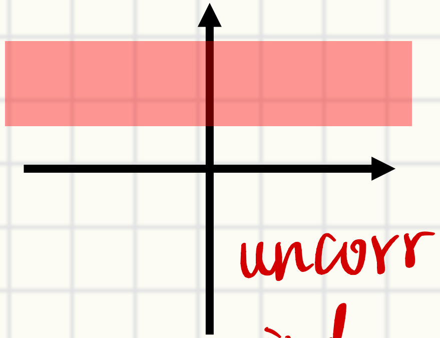
vertical / horizontal symme

(a) ✓



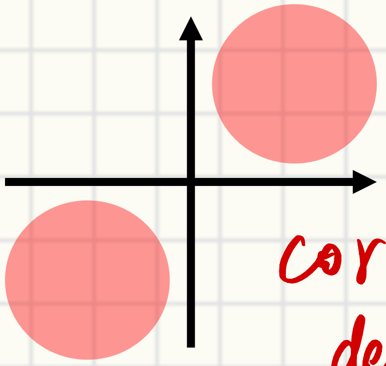
uncorr. dep.

(b) ✓



uncorr. indep.

(c)



corr dep.

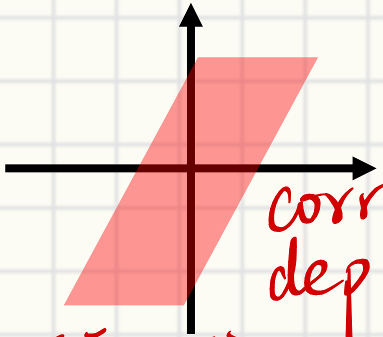
$\mu_X=\mu_Y=0$, $E[XY]>0$

$E[XY]-\mu_X\mu_Y>0$.

(d) ✓

indep, uncorr.

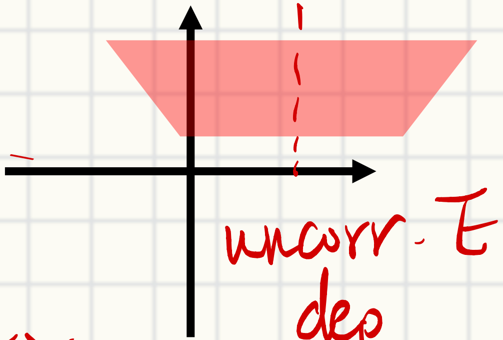
(e)



corr, dep,

$E[XY]>0$. $Cov(X,Y)>0$

(f) ✓



uncorr. $E[Y(X-\mu_X)]=0$.

dep.

Example

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY] - \mu_X\mu_Y$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

Simplify the following expressions:

- $Cov(8X + \cancel{3}, 5Y - \cancel{2})$ $= Cov(8X, 5Y) = 40\, Cov(X,Y)$
- $Cov(10X - \cancel{5}, -3X + \cancel{15})$ $= Cov(10X, -3X) = -30\, Var(X)$
- $Cov(X + \cancel{2}, 10X - 3Y)$ $= 10\, Var(X) - 3\, Cov(X,Y)$
- $\rho_{10X, Y+4}$ $= \dfrac{Cov(10X, Y+\cancel{4})}{\sigma_{10X}\ \sigma_{Y+\cancel{4}}} = \dfrac{\cancel{10}\, Cov(X,Y)}{\cancel{10}\,\sigma_X\,\sigma_Y}$ $= \rho_{X,Y}$

Example

$E\left[(X-\mu_X)(X-\mu_X)^T\right]$

X

$Var(X_1)$ $\quad E[(X_1-\mu_{X_1})(X_2-\mu_{X_2})] = Cov(X_1, X_2)$

R_X

Suppose the covariance matrix of RV vector $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ is $\begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

$\Rightarrow R_X[i,j] = Cov(X_i, X_j)$

- Find $Cov(X_1 + X_2, X_1 + X_3)$

  $= Var(X_1) + Cov(X_2, X_1) + Cov(X_1, X_3) + Cov(X_2, X_3)$

  $= 5 + 2 + 0 + 2 = 9$

- Find a s.t. $X_2 - aX_1$ is uncorrelated with X_1

- Find ρ_{X_1,X_2} $= \dfrac{Cov(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}} = \dfrac{2}{\sqrt{5}\sqrt{5}} = \dfrac{2}{5}$

- Find $Var(X_1 + X_2 + X_3)$

  $= Cov(X_1+X_2+X_3, X_1+X_2+X_3) = \sum_{i,j} R_X[i,j] = 5\times 3 + 2\times 4 = 23$

$$\mathrm{Cov}(X_2 - aX_1, X_1) = 0$$

$$\mathrm{Cov}(X_2, X_1) - a\,\mathrm{Var}(X_1) = 0$$

$$2 - a \times 5 = 0 \qquad a = 0.4$$

Sample Mean and Variance

Suppose $X_1 \ldots X_n$ are independent and identical distributed(i.i.d.) RVs with unknown mean μ and variance σ^2 (May not be Gaussian)

- Estimate μ and σ^2 by   Example → lottery

  samples. $X_1 \ldots X_n$.

- $\hat{X} = \frac{1}{n}\sum_{k=1}^{n} X_k$.
- $\widehat{\sigma^2} = \frac{1}{n-1}\sum_{k=1}^{n} (X_k - \hat{X})^2$

  $E[(X-\mu)^2]$

- Unbiased: $E[\hat{Y}] = Y$
- Is sample mean and sample variance unbiased?

$$E[\hat{X}] = E\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] = \frac{1}{n}\sum_{k=1}^{n} E[X_k]$$

$$= \frac{1}{n}\sum_{k=1}^{n} \mu = \mu$$

generality

$$E[(X_k - \hat{X})^2] \overset{k=1}{=} E[(X_1 - \hat{X})^2] = Var(X_1 - \hat{X})$$

$$= Var\left(\frac{(n-1)X_1}{n} - \sum_{k=2}^{n} \frac{X_k}{n}\right)$$

$$= \left(\frac{(n-1)^2}{n^2} + \sum_{k=2}^{n} \frac{1}{n^2}\right)\sigma^2 = \frac{(n-1)}{n}\sigma^2$$

$$E[\hat{\sigma}^2] = \frac{n}{n-1}\,\frac{n-1}{n}\,\sigma^2 = \sigma^2$$

Minimum Mean Square Error Estimation

Constant Estimator

Mean Square Error (MSE)

Given RV Y, if we know f_Y

- Constant estimator: constant $\delta^* = argmin_\delta E[(Y-\delta)^2]$
- $E[(Y-\delta)^2] = E[Y^2] - 2\delta E[Y] + \delta^2$
- Taking derivative w.r.t δ $\quad 2\delta - 2E[Y] = 0$

  $\Rightarrow \delta^* = E[Y]$

- $\delta^* = \mu_Y$

- MSE $E[(Y-\delta^*)^2] = E[Y^2] - 2\mu_Y E[Y] + \mu_Y^2$

  $= E[Y^2] - \mu_Y^2 = Var(Y)$

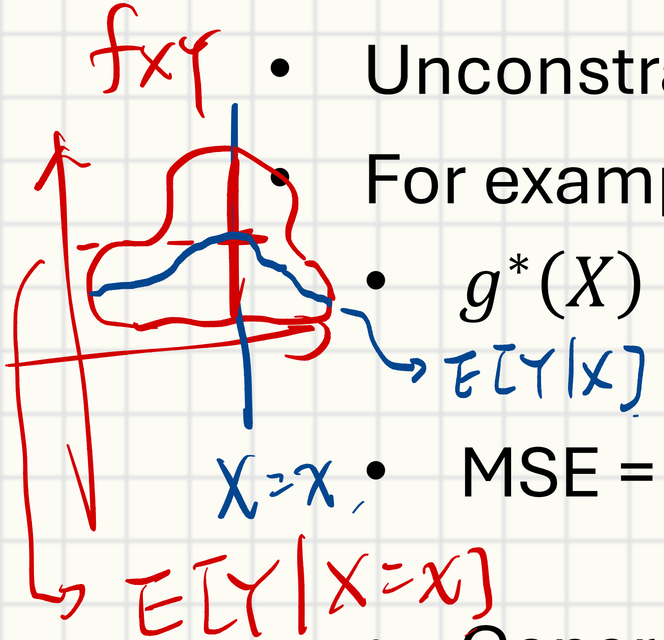
Unconstrained Estimator

Given RV X, Y, if we know $f_{X,Y}$ and some observations X

- Unconstrained Estimator - $g^*(X) = argmin_g\, E\left[\left(Y - g(X)\right)^2\right]$
- For example, say $X = 10$
  - $g^*(X) = E[Y|X = 10] = \int_{-\infty}^{\infty} v f_{Y|X}(v|u = 10)dv$
  - MSE = $E[(\underset{Y}{Y} - \overset{g^*(X)}{\underset{\text{over } Y|X}{\boxed{E[Y|X = u]}}})^2|X = u] = E[Y^2|X=u] - (\underbrace{E[Y|X=u]}_{g^*(u)})^2$
- General form

  $g^*(x) = E[Y|X]$

- MSE $E[(Y - E[Y|X])^2] = E[Y^2|X] - E_X[\underbrace{(E[Y|X])^2}_{RV}]$



Linear Estimator

What if we do not know $f_{X,Y}$ or $f_{Y|X}$ is hard to compute?

- Let $g^*(X) = aX + b$, find $argmin_{(a,b)}\, E\left[\left(Y - (aX + b)\right)^2\right]$
- Can also be written as $argmin_{(a,b)}\, E\left[\left((Y - aX) - b\right)^2\right]$
  - b is the constant estimator of $Y - aX$
  - $b =$ $E[Y - aX] = \mu_Y - a\mu_X$

Linear Estimator

$b = \mu_Y - a\mu_X$

What if we do not know $f_{X,Y}$ or $f_{Y|X}$ is hard to compute?

- Let $g^*(X) = aX + b$, find $argmin_{(a,b)}\, E[(Y - (aX + b))^2]$
- MSE $E\left[(Y - \mu_Y - a(X - \mu_X))^2\right] = Var(Y - aX) =$

$g^*(X)$

$Var(Y) - 2a\,Cov(X,Y) + a^2\,Var(X)$

- $\hat{E}[Y|X] = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(X - \mu_X) = \mu_Y + \rho_{X,Y}\sigma_Y\underbrace{\left(\frac{X-\mu_X}{\sigma_X}\right)}_{\text{Standardized } X}$

σ_X

$a^* = \frac{Cov(X,Y)}{Var(X)}$

- MSE = $\sigma_Y^2 - \frac{(Cov(X,Y))^2}{Var(X)} = \sigma_Y^2(1 - \rho_{X,Y}^2)$