Last lecture

Independent RV (Ch 4.4)

- Product Set
- Examples

Sums of joint RVs (Ch 4.5)

- Motivation
- Examples

More examples on joint RVs (Ch 4.6)

- Max of two RVs
- Buffon’s needle problems

Agenda

Independent RV (Ch 4.4)

- Product Set
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More examples on joint RVs

Max of two RVs

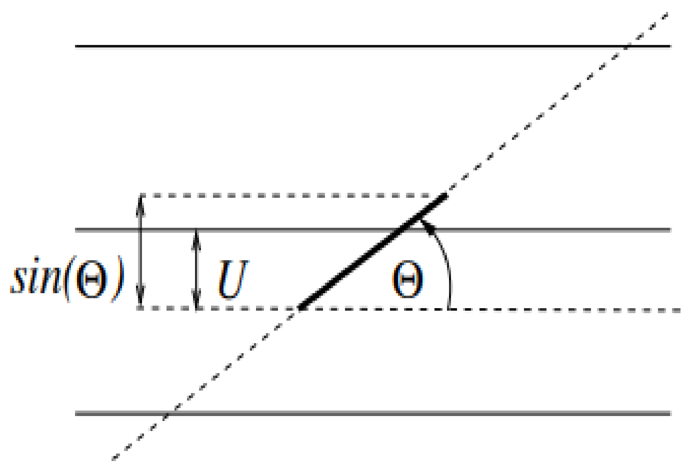
Let $W = \max(X, Y)$

- $F_W(t) = P\{W \le t\} =$
- $f_W(t) = \frac{dF_W(t)}{dt} =$

Abstract – on $P\{W \in (t, t+h]\} = f_W(t)h + o(h)$

- Case (a): $Y \le t, X \in (t, t+h]$
- Case (b): $X \le t, Y \in (t, t+h]$
- Case (c): $X \in (t, t+h], Y \in (t, t+h]$

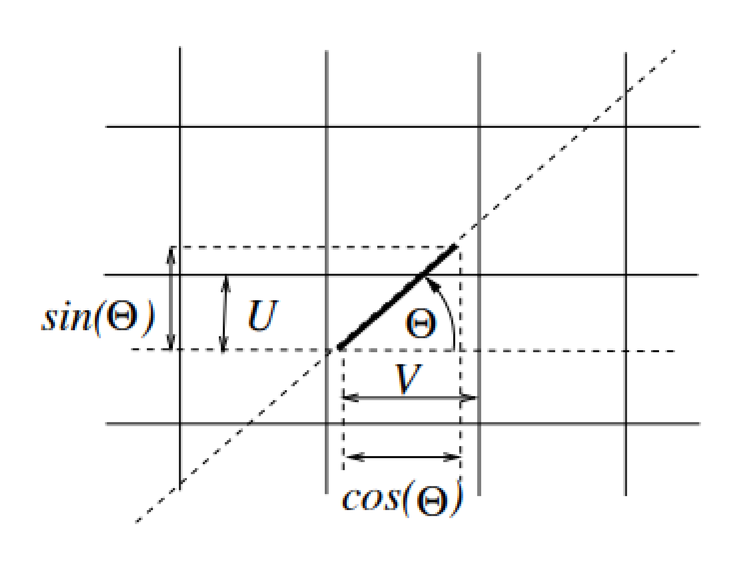
Buffon’s needle problem



- Draw many parallel horizontal lines
  - Space 1 inch between two lines
  - Throw a needle of 1 inch length on the plane
  - Find $P\{$ “The needle intersect with a line“ $\}$

Define U = “distance from the needle lower end to the first line above

Buffon’s needle problem (2)



- What if there are “horizontal” and “vertical” lines?

Let M_h denotes “missing horizontal lines”
M_v denotes “missing vertical lines”

Joint pdfs of functions of RV

Notation and Definition

Denote the point on the plane (X, Y) as a column vector $\begin{pmatrix} X \\ Y \end{pmatrix}$

- $f_{X,Y}(u, v)$ is denoted as $f_{X,Y}\left(\begin{pmatrix} u \\ v \end{pmatrix}\right)$

Suppose $W = aX + bY$ and $Z = cX + dY$

- $\begin{pmatrix} W \\ Z \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix}$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- For any $\begin{pmatrix} X \\ Y \end{pmatrix}$ in $u - v$ plane, we can find $\begin{pmatrix} W \\ Z \end{pmatrix}$ in $\alpha - \beta$ plane

Determinant and Inverse

$\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}$

- $\det(A) = ad - cb$. If $\det(A) \neq 0$
  - $\alpha - \beta$ span a plane

  - $\begin{pmatrix} u \\ v \end{pmatrix} = A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, where $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

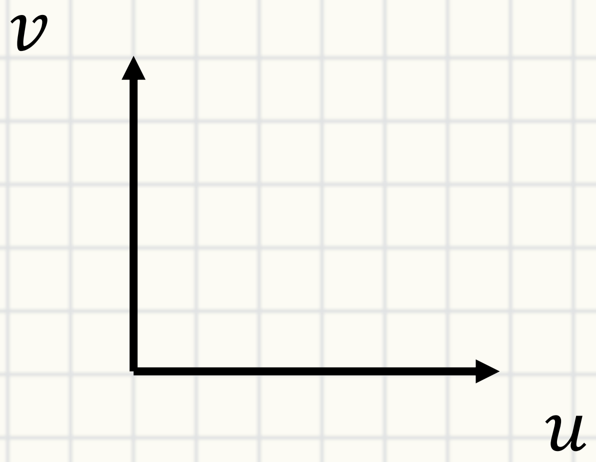
  - $|\det(A)|$ is like “ “

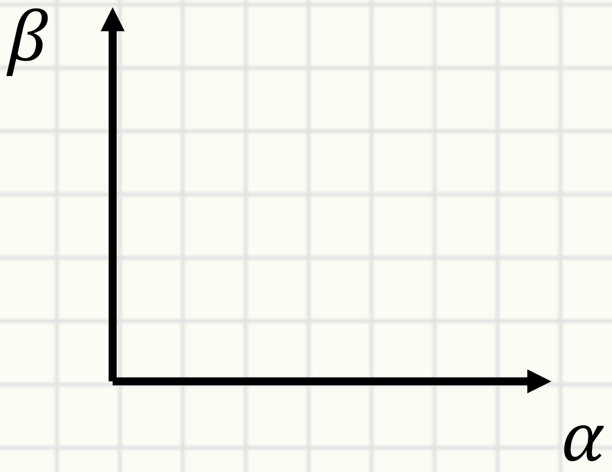
Joint PDF properties

Suppose $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$ where $\det(A) \neq 0$

- $f_{W,Z}(\alpha, \beta) = \frac{1}{|\det(A)|} f_{X,Y}(u, v)$

- Intuition - $W = 2X + Y, Z = X - Y$





Example

$W = X - Y, Z = X + Y$. Express $f_{W,Z}(\alpha, \beta)$ in terms of $f_{X,Y}$

- $\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$

- $Det(A) =$

- For $(W, Z) = (\alpha, \beta)$, $(X, Y) =$

- $f_{W,Z}(\alpha, \beta) =$

Example

Suppose X and Y are continuous independent RVs.

- $W = X + Y, Z = Y$
- Find$f_{W,Z}(\alpha, \beta)$ and $f_W(\alpha)$

Generalize to one-to-one mapping (not in exam)

Suppose $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$ where $\det(A) \neq 0$

- $f_{W,Z}(\alpha, \beta) = \frac{1}{|\det(A)|} f_{X,Y}(u, v)$

Suppose $\begin{pmatrix} W \\ Z \end{pmatrix} = g\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right) = \begin{pmatrix} g_1(x, y) \\ g_2(x, y) \end{pmatrix}$

- $f_{W,Z}(\alpha, \beta) = \frac{1}{|\mathcal{J}|} f_{X,Y}(u, v)$
- $\mathcal{J} = \begin{bmatrix} \frac{\partial g_1(u,v)}{\partial u} & \frac{\partial g_1(u,v)}{\partial v} \\ \frac{\partial g_2(u,v)}{\partial u} & \frac{\partial g_2(u,v)}{\partial v} \end{bmatrix}$

Correlation and covariance

Definition

Metrics such as mean/ variance is more convenient than PDF/ CDF

- Recall $\mu_X = E[X], Var(X) = E[(X - \mu_X)^2]$
- For jointly distributed X and Y
  - Covariance $Cov(X, Y) =$
  - Correlation coefficient $\rho_{X,Y} =$
    - $\rho_{X,Y} \in$
  - Cross moment $E[XY]$ (less used)

Definition

- $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

  $=$

- **Uncorrelated** -
- $\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$
  - $\uparrow$: X and Y has the same trend
  - $\rho_{X,Y} > 0$: Positively correlated

Properties

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY] - \mu_X\mu_Y$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

Some properties for independent and uncorrelated

- Independent uncorrelated
  - $E[XY] =$
- Uncorrelated independent
- Multiple RVs are uncorrelated if they are pairwise uncorrelated

Properties

$$Cov(X,Y) = E[(X - \mu_X)(Y - \mu_Y)]$$
$$= E[XY] - \mu_X \mu_Y$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

Some properties for independent and uncorrelated

- $Cov(X + Y, U + V) = Cov(X,U) + Cov(X,V) + Cov(Y,U) + Cov(Y,V)$
- $Cov(aX + b, cY + d) = acCov(X,Y)$
- If X and Y are independent
  - $Var(X + Y) = Cov(X + Y, X + Y) = Cov(X,X) + Cov(Y,Y)$

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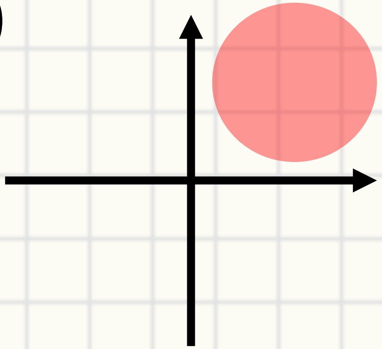
Select those are uncorrelated

(a)



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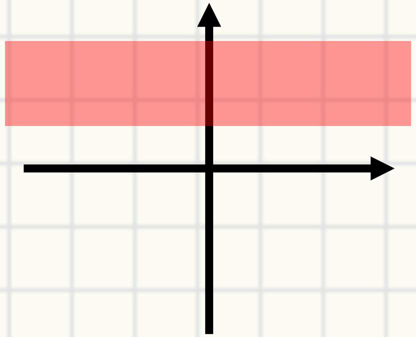

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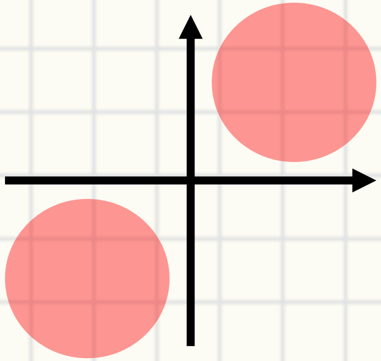

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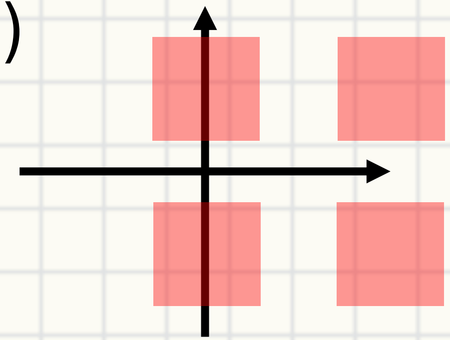

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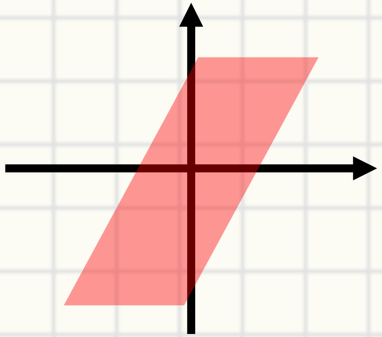

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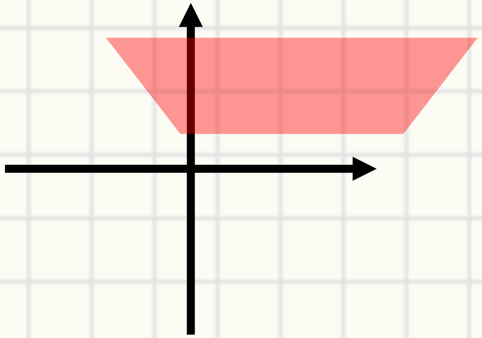

Example

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY] - \mu_X\mu_Y$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

Simplify the following expressions:

- $Cov(8X+3, 5Y-2)$
- $Cov(10X-5, -3X+15)$
- $Cov(X+2, 10X-3Y)$
- $\rho_{10X, Y+4}$

Example

Suppose the covariance matrix of RV vector $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ is $\begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

- Find $Cov(X_1 + X_2, X_1 + X_3)$
- Find a s.t. $X_2 - aX_1$ is uncorrelated with X_1
- Find ρ_{X_1,X_2}
- Find $Var(X_1 + X_2 + X_3)$