Last lecture

Independent RV (Ch 4.4)

- Product Set
- Examples

Sums of joint RVs (Ch 4.5)

- Motivation
- Examples

More examples on joint RVs (Ch 4.6)

- Max of two RVs
- Buffon’s needle problems

Agenda

Independent RV (Ch 4.4)

- Product Set
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More examples on joint RVs

Max of two RVs

Let $W = \max(X, Y)$

- $F_W(t) = P\{W \le t\} =$ 1. $X > Y \Rightarrow F_{X,Y}(t, \ )$ 2. $Y > X \Rightarrow F_{X,Y}(\ , t)$ $= F_{X,Y}(t, t)$ $= F_X(t)F_Y(t)$ if X, Y are ind.

- $f_W(t) = \frac{dF_W(t)}{dt} =$

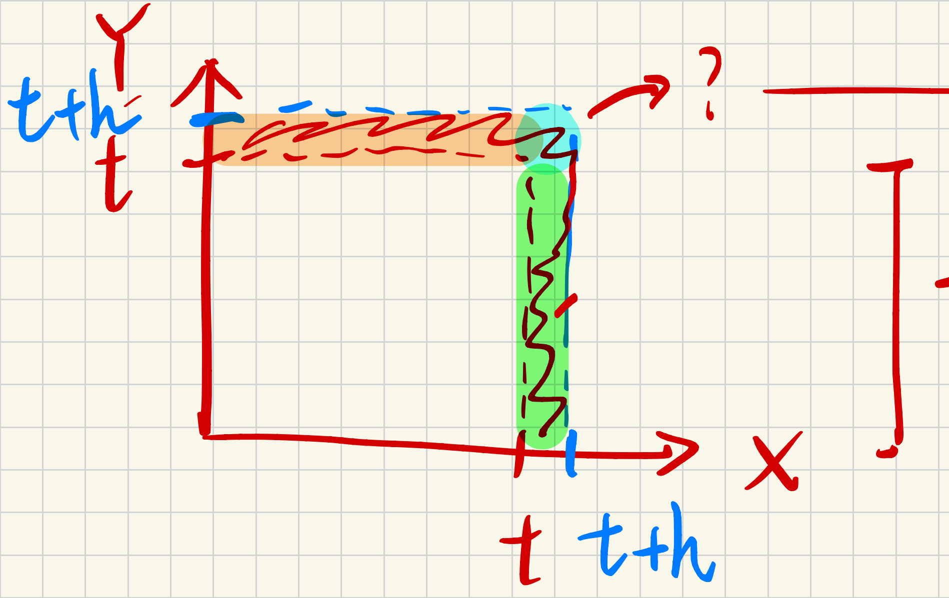
Abstract – on $P\{W \in (t, t+h]\} = f_W(t)h + o(h)$

- Case (a): $Y \le t, X \in (t, t+h]$
- Case (b): $X \le t, Y \in (t, t+h]$
- Case (c): $X \in (t, t+h], Y \in (t, t+h]$

$f_W(t) = \frac{dF_X(t)}{dt} F_Y(t) + F_X(t) \frac{dF_Y(t)}{dt}$ ($\frac{dF_X(t)}{dt} = f_X(t)$, $\frac{dF_Y(t)}{dt} = f_Y(t)$)

if X, Y are independ. $W = \max(X, Y)$

$$f_W(t) = f_X(t) F_Y(t) + F_X(t) f_Y(t)$$



$\to f_X(t) f_Y(t)$

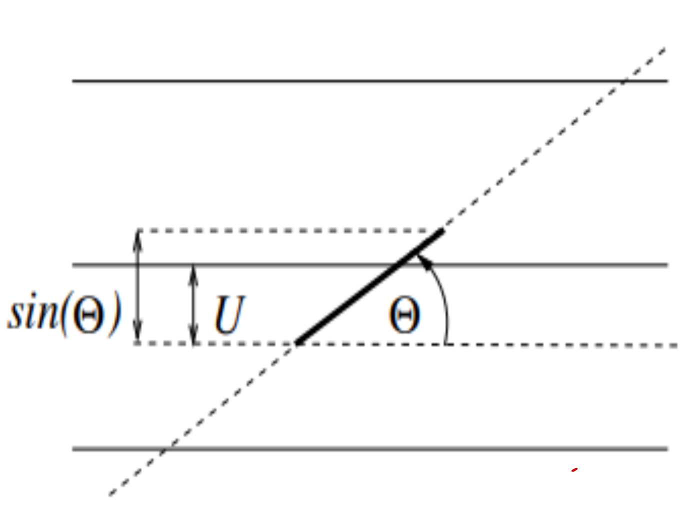
$\sqcup\ f_X(t)$

Alter. $F_W^c(t)$

$$= [1 - F_X(c)][1 - F_Y(c)]$$

$X > t$ $\quad$ $Y > t$

Buffon’s needle problem

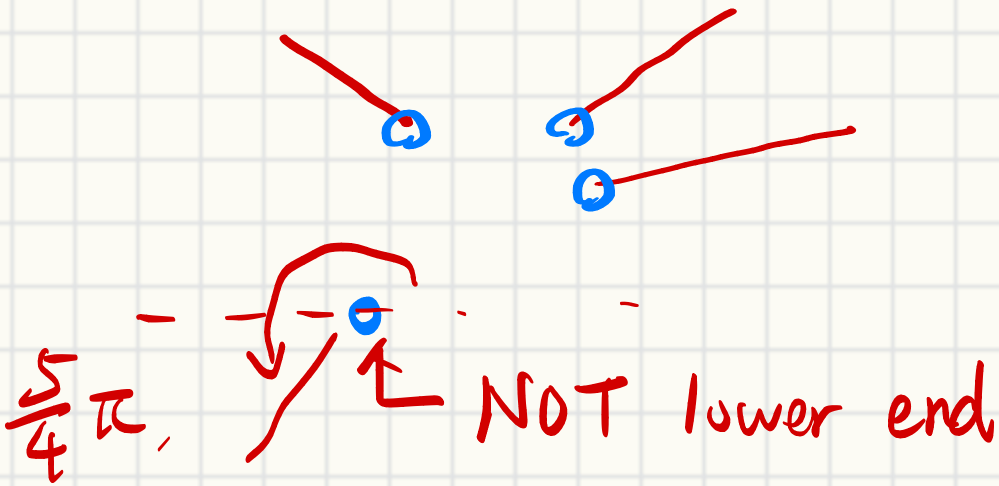


- Draw many parallel horizontal lines
    - Space 1 inch between two lines
    - Throw a needle of 1 inch length on the plane
    - Find P{ “The needle intersect with a line“ } I.

Define U = “distance from the needle lower end to the first line above

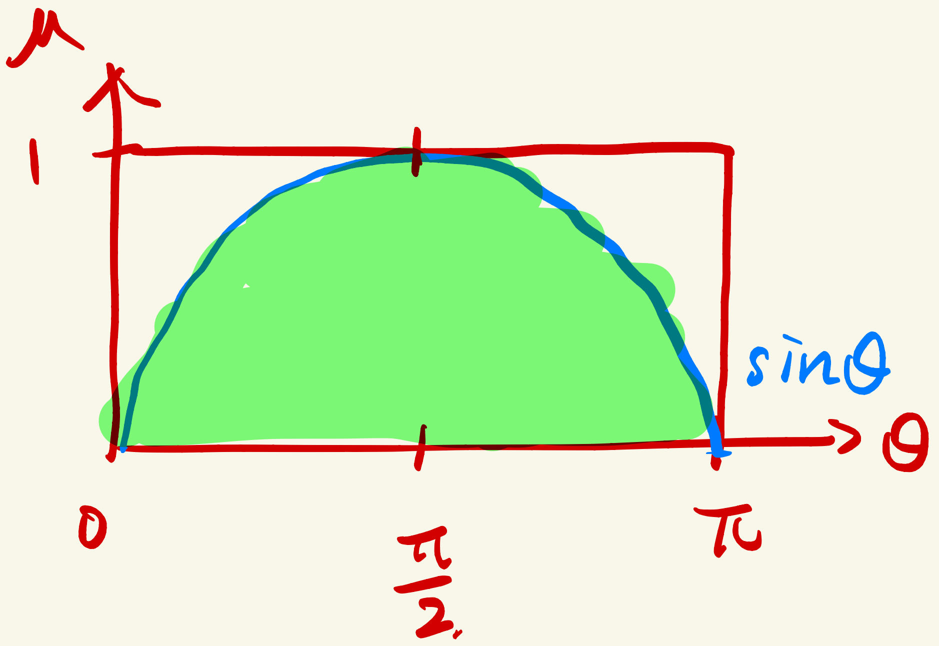
$U \sim \text{Uniform}[0,1]$

$\Theta \sim \text{Uniform}[0,\pi]$



$$P(I) = P\{U \le \sin\Theta\}$$

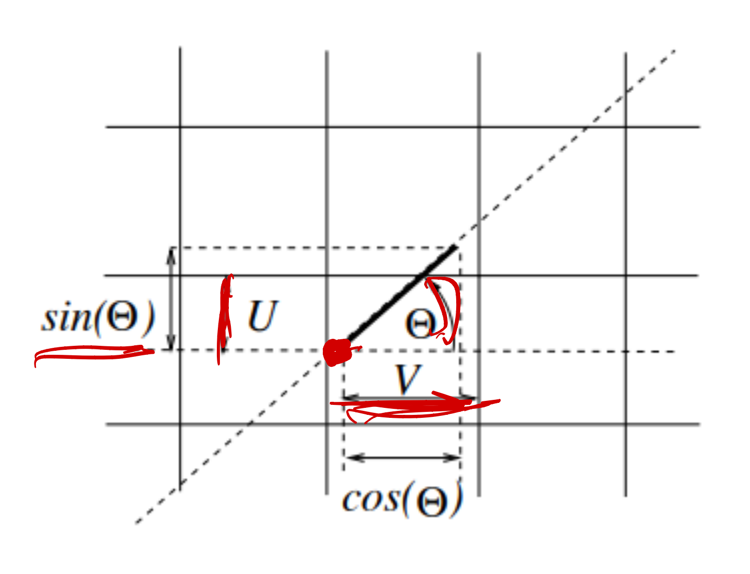
$$= \int_0^{\pi}\int_0^{\sin\theta} \underbrace{f_{U,\Theta}(u,\theta)}_{\frac{1}{\pi}}\, du\, d\theta$$



$$= \int_0^{\pi} \frac{\sin\theta}{\pi}\, d\theta = \left[-\frac{\cos\theta}{\pi}\right]_0^{\pi} = \frac{2}{\pi}$$

Buffon's needle problem (2)

- What if there are "horizontal" and "vertical" lines?



Let M_h denotes "missing horizontal lines"
M_v denotes "missing vertical lines"

1 - cos Θ

$P(M_h \cap M_v)$

Wrong guess - M_h & M_v are independent

$P(M_h \cap M_v) = \left(1 - \frac{2}{\pi}\right)^2 \approx 0.132$ X False !

$$P(M_h \cap M_v) = \int_0^{\pi} \underline{\underline{(1-\sin\theta)}} \left(1 - |\cos\theta|\right) . \, du \, d\theta$$

$$= 1 - \frac{3}{\pi} \simeq 0.045$$

Joint pdfs of functions of RV

Notation and Definition

Denote the point on the plane (X, Y) as a column vector $\begin{pmatrix} X \\ Y \end{pmatrix}$

- $f_{X,Y}(u, v)$ is denoted as $f_{X,Y}\left(\begin{pmatrix} u \\ v \end{pmatrix}\right)$

Suppose $W = aX + bY$ and $Z = cX + dY$

- $\begin{pmatrix} W \\ Z \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix}$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- For any $\begin{pmatrix} X \\ Y \end{pmatrix}$ in $u - v$ plane, we can find $\begin{pmatrix} W \\ Z \end{pmatrix}$ in $\alpha - \beta$ plane

$X = u, Y = v$. $\quad W = \alpha, Z = \beta$

Determinant and Inverse

$$\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}, \text{ where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}$

- $\det(A) = ad - cb$. If $\det(A) \neq 0$
  - $\alpha - \beta$ span a plane

  - $\begin{pmatrix} u \\ v \end{pmatrix} = A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, where $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

  - $|\det(A)|$ is like " scaling factor "

Joint PDF properties

Suppose $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$ where $\det(A) \neq 0$

- $f_{W,Z}(\alpha, \beta) = \frac{1}{|\det(A)|} f_{X,Y}(u, v)$

- Intuition - $W = 2X + Y, Z = X - Y$

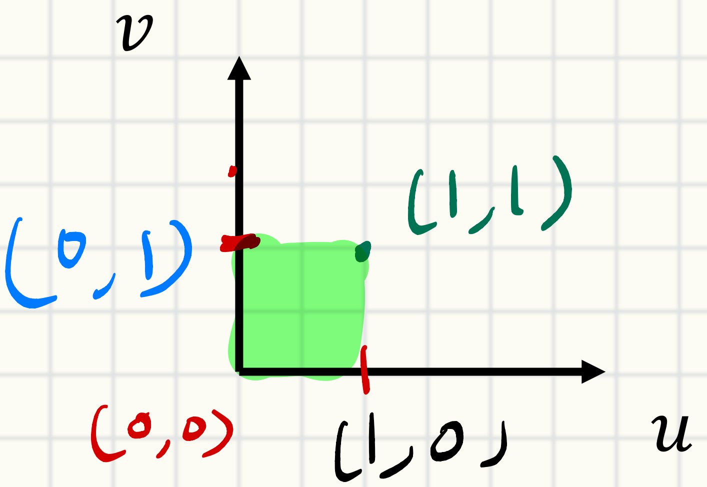
$X, Y \sim \text{Uniform}[0,1]$

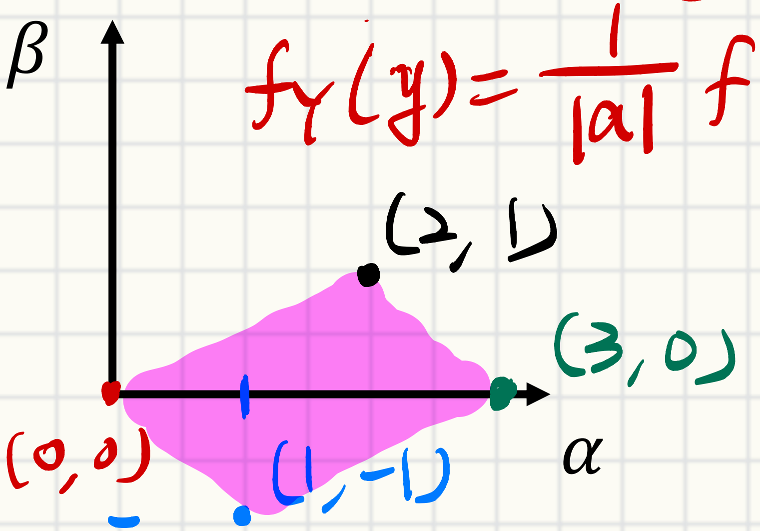
Note: if $Z = 1$, $Y = 1$.

$W = Y \Rightarrow Y = aX + b$.

$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$

$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$





Example

$W = X - Y, Z = X + Y$. Express $f_{W,Z}(\alpha, \beta)$ in terms of $f_{X,Y}$

- $\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$
- $Det(A) =$ $(1 \times 1) - (-1 \times 1) = 2$
- For $(W, Z) = (\alpha, \beta)$, $(X, Y) =$ $\left(\frac{\alpha+\beta}{2}, \frac{-\alpha+\beta}{2}\right)$
- $f_{W,Z}(\alpha, \beta) =$

$\frac{1}{|2|} \times f_{XY}\left(\frac{\alpha+\beta}{2}, \frac{-\alpha+\beta}{2}\right)$

$W = X - Y$

$+) \; Z = X + Y$

$W + Z = 2X$

$X = \frac{W+Z}{2}$

$Y = \frac{-W+Z}{2}$

Example

Suppose X and Y are continuous independent RVs.

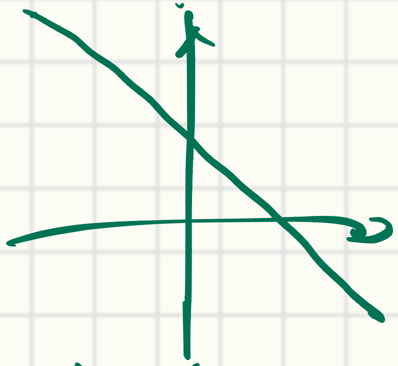
- $W = X + Y, Z = Y$
- Find $f_{W,Z}(\alpha, \beta)$ and $f_W(\alpha)$

$$\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\det(A) = 1 \times 1 - 0 = 1$$

$$\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \alpha-\beta \\ \beta \end{pmatrix}$$

$S = X + Y$



$$f_{W,Z}(\alpha, \beta) = \frac{1}{|1|} \times f_{XY}(\alpha-\beta, \beta)$$

$$f_W(\alpha) = \int_{-\infty}^{\infty} f_{XY}(\alpha-\beta, \beta)\, d\beta = f_S(c) \text{ in Ch 3.6}$$

Generalize to one-to-one mapping (not in exam)

Suppose $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$ where $\det(A) \neq 0$

- $f_{W,Z}(\alpha, \beta) = \frac{1}{|\det(A)|} f_{X,Y}(u, v)$

⇒ det(A) is const $\forall(\alpha, \beta)$

Suppose $\begin{pmatrix} W \\ Z \end{pmatrix} = g\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right) = \begin{pmatrix} g_1(x, y) \\ g_2(x, y) \end{pmatrix}$

e.g. $W = X^2$.

⇒ $\begin{pmatrix} u \\ v \end{pmatrix} = g^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

counter example.

$Z = |X|$

- $f_{W,Z}(\alpha, \beta) = \frac{1}{|\mathcal{J}|} f_{X,Y}(u, v)$ → $|\det(\mathcal{J})|$
- $\mathcal{J} = \begin{bmatrix} \frac{\partial g_1(u,v)}{\partial u} & \frac{\partial g_1(u,v)}{\partial v} \\ \frac{\partial g_2(u,v)}{\partial u} & \frac{\partial g_2(u,v)}{\partial v} \end{bmatrix}$

↪ $|\det(\mathcal{J})| = f(\alpha, \beta)$

Correlation and covariance

Definition

Metrics such as mean/ variance is more convenient than PDF/ CDF

- Recall $\mu_X = E[X], Var(X) = E[(X - \mu_X)^2]$
- For jointly distributed X and Y
  - Covariance $Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$
    $= E[XY] - \mu_X \mu_Y$
  - Correlation coefficient $\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$
    - $\rho_{X,Y} \in [-1, 1]$
  - Cross moment $E[XY]$ (less used)

(oftenly used as "cross correlation" in DSP)
(func xcorr)

Definition

- $Cov(X,Y) = E[(X - \mu_X)(Y - \mu_Y)]$
  $= E[XY - \mu_X Y - X\mu_Y + \mu_X\mu_Y]$
  $= E[XY] - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y$
  $= E[XY] - \mu_X\mu_Y$
- **Uncorrelated** - $Cov(X,Y) = 0$
- $\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$
  - $\uparrow$: X and Y has the same trend
  - $\rho_{X,Y} > 0$: Positively correlated

Properties

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY] - \mu_X\mu_Y$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

Some properties for independent and uncorrelated

- Independent implies. uncorrelated
  - $E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uv f_{XY}(u,v)\,du\,dv = \int_{-\infty}^{\infty} u f_X(u)\,du \times \int_{-\infty}^{\infty} v f_Y(v)\,dv$
  $= E[X]E[Y]$
  $= \mu_X \mu_Y$
- Uncorrelated does not imply independent
- Multiple RVs are uncorrelated if they are pairwise uncorrelated

Properties

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY] - \mu_X\mu_Y$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

Some properties for independent and uncorrelated

- $Cov(X+Y, U+V) = Cov(X,U) + Cov(X,V) + Cov(Y,U) + Cov(Y,V)$

- $Cov(aX+b, cY+d) = acCov(X,Y)$

- If X and Y are independent
  - $Var(X+Y) = Cov(X+Y, X+Y) = Cov(X,X) + Cov(Y,Y)$

$= \sigma_X^2 + \sigma_Y^2$