Last lecture

Binary Hypothesis Testing with continuous distribution (Ch 3.10)

- Example

Jointly Distributed RV/ Joint CDF (Ch 4.1)

- Motivation
- Definition
- Properties

Joint PMF (Ch 4.2)

- Definition
- Example

Agenda

Joint PDF (Ch 4.3)

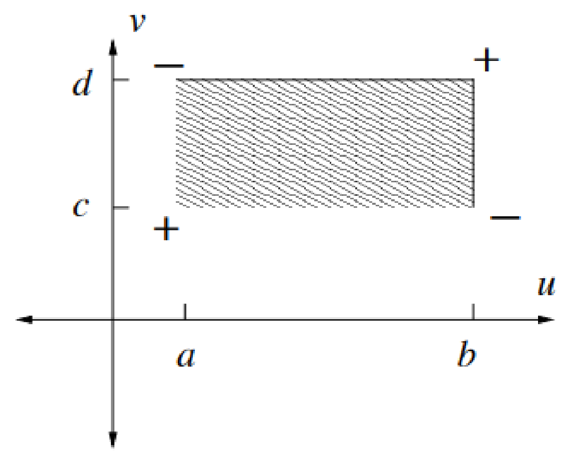
- Definition
- Examples
    - Uniform distribution
    - Conditional distribution

Independent RV (Ch 4.4)

- From event to RV - CDF
- Check using PDF

Joint PDF

Joint PDF



If X and Y are continuous, joint PDF $f_{X,Y}(u,v)$ s.t.

- $F_{X,Y}(u,v) = \int_{-\infty}^{u} \int_{-\infty}^{v} f_{X,Y}(u,v) dv du$
- $P\{(X,Y) \in R\} = \int \int_R f_{X,Y}(u,v) du dv$ for piecewise diff. R
- $f_{X,Y}(u,v) \geq 0$ for any $(u,v) \in \mathbb{R}^2$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(u,v) du dv =$

LOTUS: $E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(u,v) f_{X,Y}(u,v) du dv$

- $E[X] =$

Joint PDF

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v) f_{X,Y}(u,v)\,du\,dv$$

- $E[aX + bY + c] =$

Marginalization - Projection

- $f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv$
- $E[X] = \int_{-\infty}^{\infty} u \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv\,du$

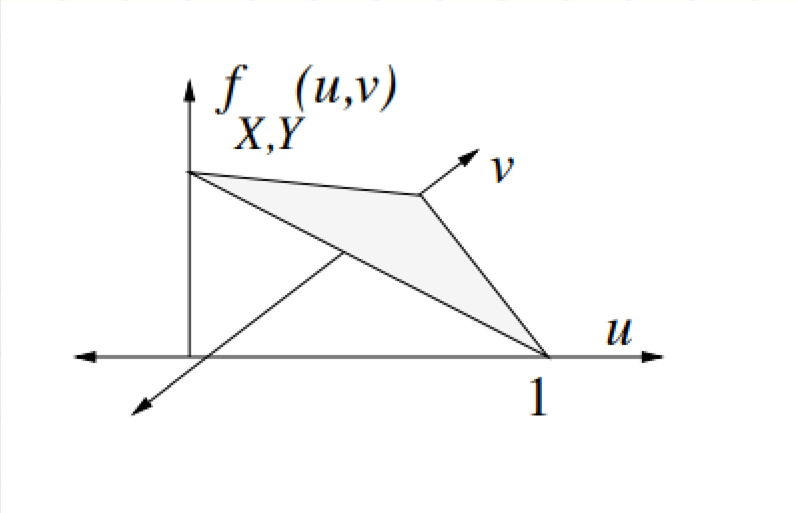
Conditional PDF - Slice

- $f_{Y|X}(v|u_0) = \frac{f_{X,Y}(u_0,v)}{f_X(u_0)}$
- Conditional expectation $E[Y|X = u_0] = \int_{-\infty}^{\infty} v f_{Y|X}(v|u_0)\,dv$

Example - Linear

Given $f_{X,Y}(u,v) = \begin{cases} c(1-u-v) & if\ u,v \geq 0, u+v \leq 1 \\ 0 & else \end{cases}$, solve

- c
- Marginal PDF $f_X(u)$
- Conditional PDF $f_{Y|X}(v|u_0)$



Uniform joint PDF

If pdf are constant over support S

- $f_{X,Y}(u,v) = \begin{cases} \frac{1}{area(S)} & if\ (u,v) \in S \\ 0 & else \end{cases}$
- S can be none-rectangular
- $P\{(X,Y) \in A\} = \frac{area(A \cap S)}{area(S)}$

Example

Let (X, Y) uniformly distributed over the uniform disk.

- $f_{X,Y}(u,v) = \begin{cases} c & if\ u^2 + v^2 \leq 1 \\ 0 & else \end{cases}$, solve
- c
- $P\{X \geq 0 \cap Y \geq 0\}$
- $P\{X^2 + Y^2 \leq r^2\}$
- $f_X(u)$
- $f_{Y|X}(v|u_0)$

Example

Let X uniformly distributed over $[0,1]$. Given $X = u$, Y is uniformly distributed over $[u, 1]$, find

- $f_{Y|X}(v|u_0)$
- $f_{XY}(u, v)$
- $f_Y(v)$

Midterm Review

Midterm Review

- Next Thursday class time
- Vote for the contents!

#2314418

Independent RV

Definition

RV X and Y are independent if for any event pairs $\{X \in A\}$ and $\{Y \in B\}$

- $P\{X \in A, Y \in B\} = P\{X \in A\} \times P\{Y \in B\}$

- Let $A = \{u: u \leq u_0\}$ and $B = \{v: v \leq v_0\}$

Even more powerful - for region $R = (a, b] \times (c, d]$

- $P\{a < X \leq b, c < Y \leq d\} = F_X(b)F_Y(d) - F_X(b)F_Y(c) - F_X(a)F_Y(d) + F_X(a)F_Y(c) =$

For PDF and CDF, independent iif

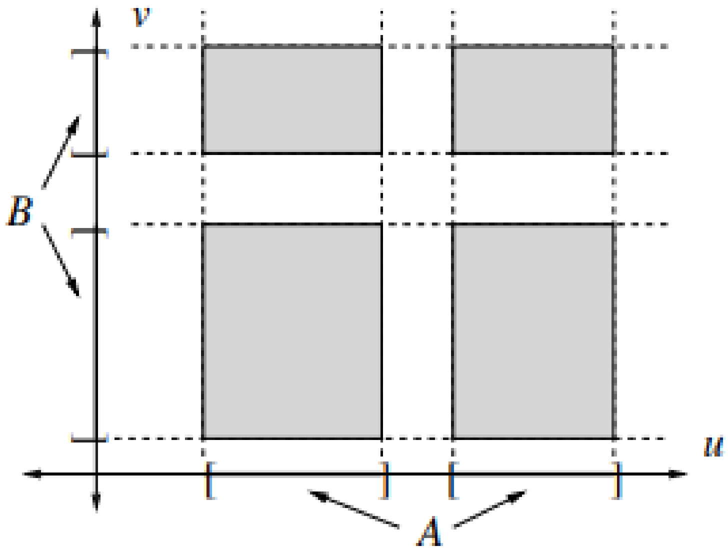
Determining independence from PDF

RV X and Y are independent if and only if

- $f_{XY}(u,v) = f_X(u)f_Y(v)$... but others?

Proposition - X and Y are independent if and only if the following condition holds: For all $u \in \mathbb{R}$, either $f_X(u) = 0$ or $f_{Y|X}(v|u) = f_Y(v)$ for all v ∈ R.

Product Set



Let A, B denote a finite union of intervals

- $|A|$ denotes the total length of $|A|$

The product set $A \times B = \{(u, v): u \in A, v \in B\}$

- The total area $|A \times B| = |A| \times |B|$

Swap property: $S \in \mathbb{R}^2$ has the swap property if

- For any pair of points $(a, b), (c, d) \in S$, (a, d) and (c, b) also in S

Proposition - $S \in \mathbb{R}^2$, S is a product set if and only if it has the swap property

Properties of independent

- If X, Y are independent and jointly continuous type RVs, then support of $f_{X,Y}$ is a product set

- Support X, Y are uniformly distributed over set $S \in \mathbb{R}^2$, then X and Y are independent iif S is a product set

Examples

Decide whether the if X and Y are independent if

- $f_{X,Y}(u,v) = Cu^2v^2$ for $u,v > 0$ and $u + v \leq 1$; 0 else
- $f_{X,Y}(u,v) = u + v$ for $u,v \in [0,1]$; 0 else
- $f_{X,Y}(u,v) = 9u^2v^2$ for $u,v \in [0,1]$; 0 else

Slido

#1245552

X and Y are independent based if $f_{XY}(u, v)$ is...