Last lecture

Binary Hypothesis Testing with continuous distribution (Ch 3.10)

- Example

Jointly Distributed RV/ Joint CDF (Ch 4.1)

- Motivation
- Definition
- Properties

Joint PMF (Ch 4.2)

- Definition
- Example

Agenda

Joint PDF (Ch 4.3)

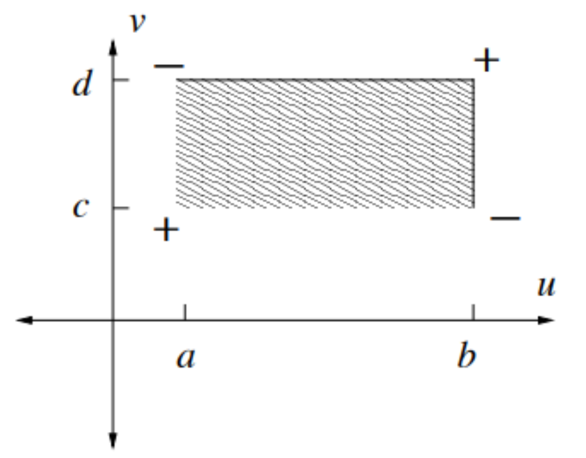
- Definition
- Examples
  - Uniform distribution
  - Conditional distribution

Independent RV (Ch 4.4)

- From event to RV - CDF
- Check using PDF

Joint PDF

Joint PDF



If X and Y are continuous, joint PDF $f_{X,Y}(u,v)$ s.t.

- $F_{X,Y}(u,v) = \int_{-\infty}^{u} \int_{-\infty}^{v} f_{X,Y}(u,v)\,dv\,du$
- $P\{(X,Y) \in R\} = \int \int_{R} f_{X,Y}(u,v)\,du\,dv$ for piecewise diff. R
- $f_{X,Y}(u,v) \geq 0$ for any $(u,v) \in \mathbb{R}^2$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,du\,dv = 1$

LOTUS: $E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(u,v) f_{X,Y}(u,v)\,du\,dv$

- $E[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u\, f_{XY}(u,v)\,du\,dv$

Joint PDF

$[au+bv+c]$

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v) f_{X,Y}(u,v)\,du\,dv$$

- $E[aX + bY + c] =$ $aE[X] + bE[Y] + c$

⇒ Expectation is linear regardless of f_{XY}.

Marginalization - Projection

- $f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv$ ⇒ Integrate over all possible Y.
- $E[X] = \int_{-\infty}^{\infty} u \underbrace{\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv}_{f_X(u)}\,du$

Conditional PDF - Slice

- $f_{Y|X}(v|u_0) = \dfrac{f_{X,Y}(u_0,v)}{f_X(u_0)}$ $\mu_{Y|X}$.
- Conditional expectation $E[Y|X = u_0] = \int_{-\infty}^{\infty} v f_{Y|X}(v|u_0)\,dv$

Example - Linear

$1-(u+v)$

Given $f_{X,Y}(u,v) = \begin{cases} c(1-u-v) & if\ u,v \geq 0, u+v \leq 1 \\ 0 & else \end{cases}$, solve

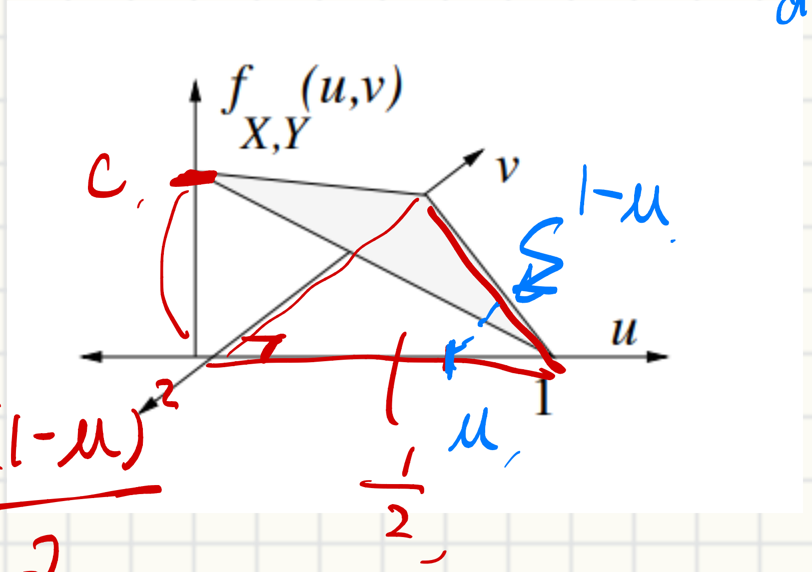
- c $\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(u,v)\,du\,dv = 1$

$= \int_0^1 \int_0^{1-v} c(1-u-v)\,du\,dv$

$= \int_0^1 \left[cu - \frac{cu^2}{2} - cuv\right]_0^{1-v} dv$

$= \int_0^1 c(1-v) - \frac{c(1-v)^2}{2} - cv(1-v)\,dv$

$AUC = 1 \times 1 \div 2 \times c/3 = 1 \quad c = 6$



- Marginal PDF $f_X(u)$

$= \int_{-\infty}^{\infty} f_{XY}(u,v)\,dv = \int_0^{1-u} c(1-u-v)\,dv = \frac{c(1-u)^2}{2}$

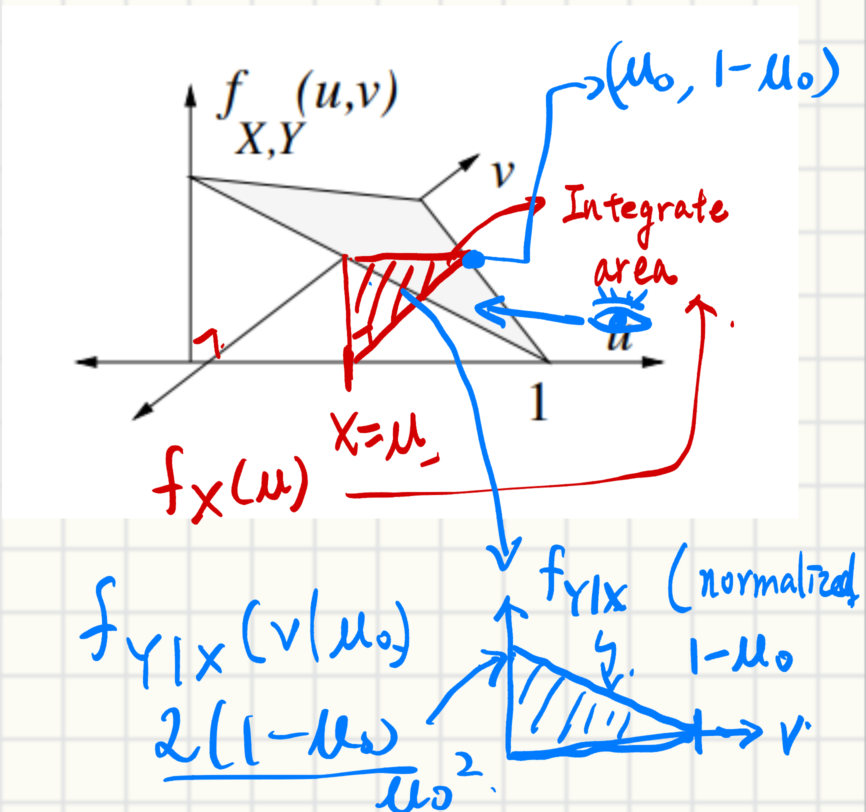
- Conditional PDF $f_{Y|X}(v|u_0)$

$f_{Y|X}(v|u_0) = \frac{f_{XY}(u_0,v)}{f_X(u_0)} = \frac{2\cancel{c}(1-u_0-v)}{\cancel{c}(1-u_0)^2} \quad 0 \leq v \leq 1-u_0$

Example - Linear

Given $f_{X,Y}(u,v) = \begin{cases} c(1-u-v) & if\ u,v \geq 0, u+v \leq 1 \\ 0 & else \end{cases}$, solve

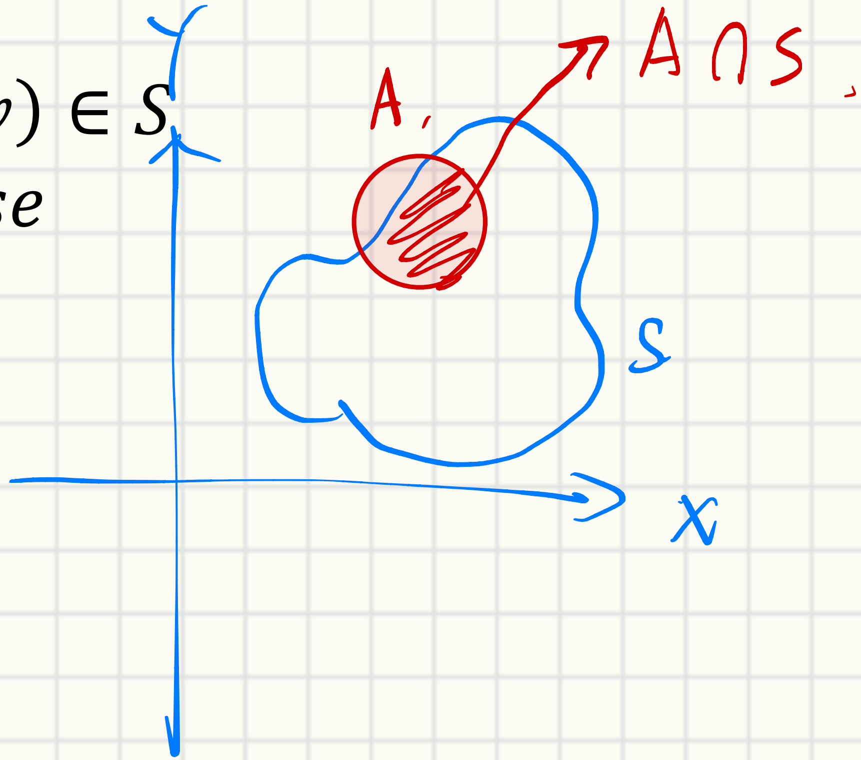
- c
- Marginal PDF $f_X(u)$
- Conditional PDF $f_{Y|X}(v|u_0)$



Uniform joint PDF

If pdf are constant over support S

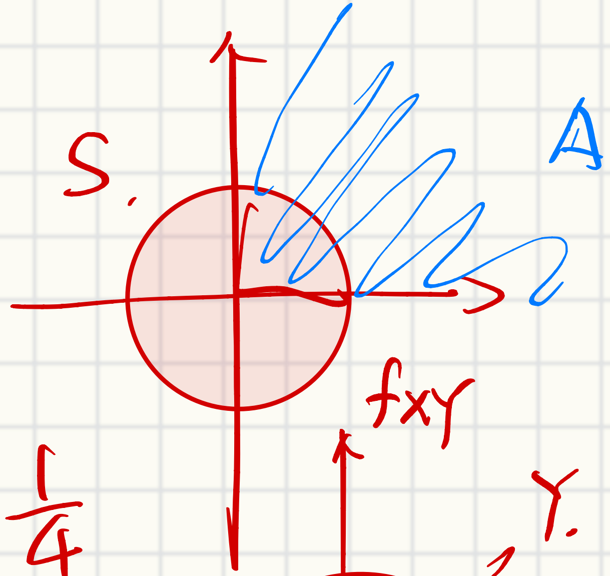
- $f_{X,Y}(u,v) = \begin{cases} \frac{1}{area(S)} & if\ (u,v) \in S \\ 0 & else \end{cases}$
- S can be none-rectangular
- $P\{(X,Y) \in A\} = \frac{area(A \cap S)}{area(S)}$

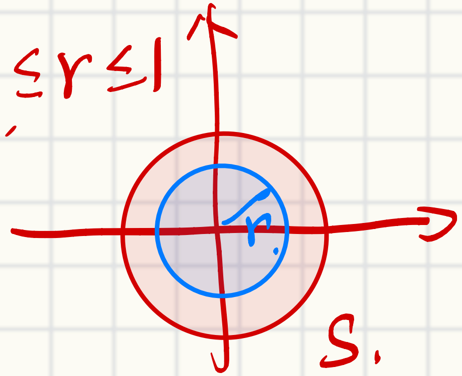


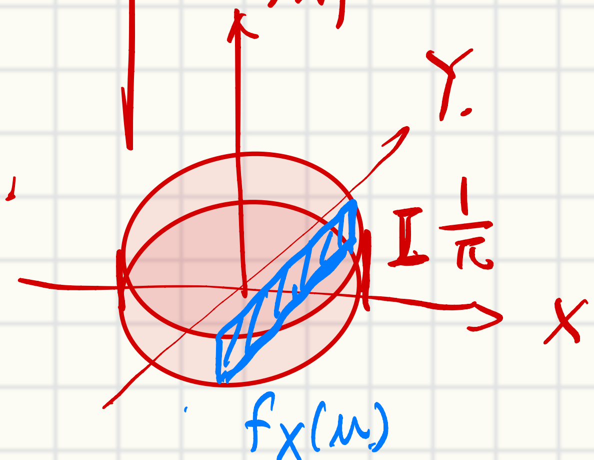
Example

Let (X, Y) uniformly distributed over the uniform disk.

- $f_{X,Y}(u,v) = \begin{cases} c & if\ u^2 + v^2 \le 1 \\ 0 & else \end{cases}$, solve $S.$
- $c = \frac{1}{Area(S)} = \frac{1}{\pi}$
- $P\{X \ge 0 \cap Y \ge 0\} = \frac{Area(A \cap S)}{Area(S)} = \frac{\frac{1}{4}\pi}{\pi} = \frac{1}{4}$
- $P\{X^2 + Y^2 \le r^2\} = \begin{cases} \frac{\pi r^2}{\pi} & if\ -1 \le r \le 1 \\ 1 & else \end{cases}$
- $f_X(u)$
- $f_{Y|X}(v|u_0)$







$$f_X(u) = \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} f_{XY}(u,v)\,dv.$$

$$u^2+v^2 \le 1.$$
$$\Downarrow$$
$$-\sqrt{1-u^2} \le v \le \sqrt{1-u^2}$$

$$= \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \frac{1}{\pi}\,dv = \left[\frac{1}{\pi}v\right]_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} = \frac{2\sqrt{1-u^2}}{\pi}$$

$$f_{Y|X}(v|u_0) = \frac{f_{XY}(u_0,v)}{f_X(u_0)} = \frac{\frac{1}{\pi}}{\quad} = \frac{1}{2\sqrt{1-u_0^2}}.$$

if $-\sqrt{1-u_0^2} \le v \le \sqrt{1-u_0^2}$

0 otherwise

Example

Let X uniformly distributed over $[0,1]$. Given $X = u$, Y is uniformly distributed over $[u, 1]$, find

- $f_{Y|X}(v|u_0)$ $= \frac{1}{1-u_0}$ if $u \le v \le 1$
- $f_{XY}(u,v)$ $= f_{Y|X}(v|u_0)\, f_X(u_0) = \frac{1}{1-u_0} \cdot 1$
- $f_Y(v)$

$p(XY) = p(Y|X)\,p(X)$ $\quad = \frac{1}{1-u}$ if $0 \le u \le v \le 1$; 0 else

$= \int_{-\infty}^{\infty} f_{XY}(u,v)\,du = \int_0^v \frac{1}{1-u}\,du = -\ln(1-v)$ if $0 < v \le 1$; 0 else

Midterm Review

Midterm Review

- Next Thursday class time
- Vote for the contents!

#2314418

Independent RV

Definition

E.g. $X \triangleq$ Roll D6

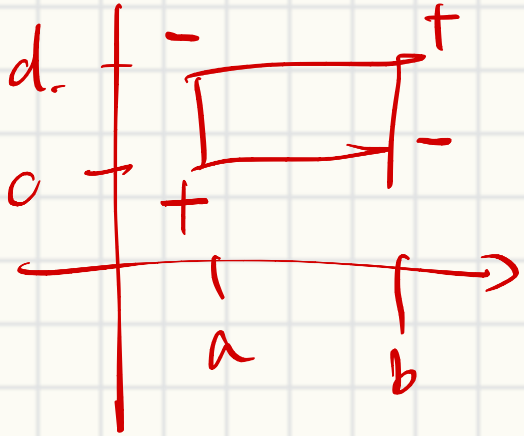
$Y \triangleq$ Toss a coin 4 times, #H.

RV X and Y are independent if for any event pairs $\{X \in A\}$ and $\{Y \in B\}$

- $P\{X \in A, Y \in B\} = P\{X \in A\} \times P\{Y \in B\}$

- Let $A = \{u: u \leq u_0\}$ and $B = \{v: v \leq v_0\}$

$\hookrightarrow F_{XY}(u_0, v_0) = F_X(u_0) \times F_Y(v_0)$



Even more powerful - for region $R = (a, b] \times (c, d]$

- $P\{a < X \leq b, c < Y \leq d\} = F_X(b)F_Y(d) - F_X(b)F_Y(c) - F_X(a)F_Y(d) + F_X(a)F_Y(c) = (F_X(b) - F_X(a))(F_Y(d) - F_Y(c))$

For PDF and ~~CDF~~ PMF, independent iif $P_{XY}(u, v) = P_X(u) P_Y(v)$

$f_{XY}(u, v) = f_X(u) f_Y(v)$

Determining independence from PDF

RV X and Y are independent if and only if

- $f_{XY}(u,v) = f_X(u)f_Y(v)$... but others?

Proposition - X and Y are independent if and only if the following condition holds: For all $u \in \mathbb{R}$, either $f_X(u) = 0$ or $f_{Y|X}(v|u) = f_Y(v)$ for all v ∈ R.

$$f_{Y|X}(v|u) = \frac{f_{XY}}{f_X} \overset{\text{indep.}}{=} \frac{\cancel{f_X} \cdot f_Y}{\cancel{f_X}}$$

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