Last lecture

Functions of a random variable (Ch 3.8)

- General form of increasing function g
- Create uniform distribution using CDF as a function

Generating a customized RV (Ch 3.8.2)

- Intuition $g = F_X^{-1}$
- Examples
  - Uniform to Exponential
  - Uniform to D6 outcome
- Area rule – Compute $E[X]$ using F_X (Ch 3.8.3)

Agenda

Binary Hypothesis Testing with continuous distribution (Ch 3.10)

- Example

Jointly Distributed RV/ Joint CDF (Ch 4.1)

- Motivation
- Definition
- Properties

Joint PMF (Ch 4.2)

- Definition
- Example

Binary Hypothesis Testing on Continuous Distribution

Overview

Similar to discrete, but with some changes

- $P\{X = u|H_1\} \to f_1(u)$
- Likelihood Ratio $\Lambda(u) = \frac{f_1(u)}{f_2(u)}$
- LRT rule $\Lambda(X) \begin{cases} > \tau & H_1 \\ < \tau & H_0 \end{cases}$

$p_{false\ alarm}, p_{miss}, p_e$ remain the same

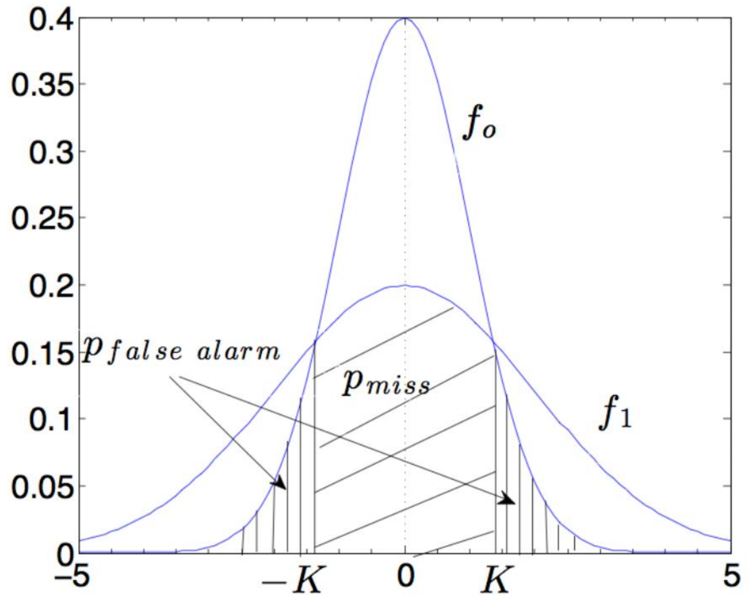
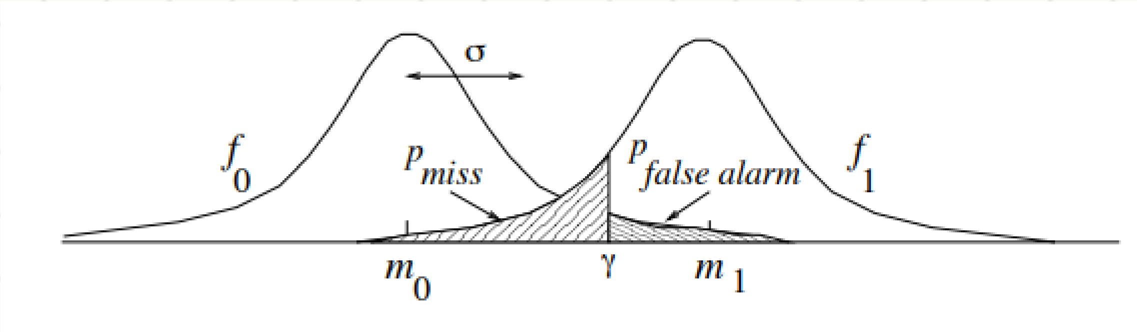


Figure 3.27: $N(0,1)$ and $N(0,4)$ pdfs and ML threshold K.

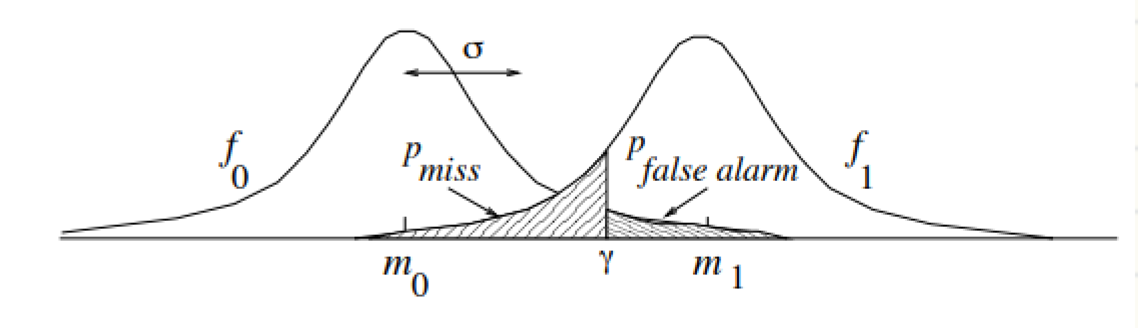
Example



X under H_i follows $N(m_i, \sigma^2)$. Given m_i, σ, π_i, Find ML and MAP rule

- $f_i(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(u-m_i)^2}{2\sigma^2}\right\}$
- $\Lambda(u) = \exp\{(u - \frac{m_0+m_1}{2})(\frac{m_1-m_0}{\sigma^2})\}$
- ML rule –

Example



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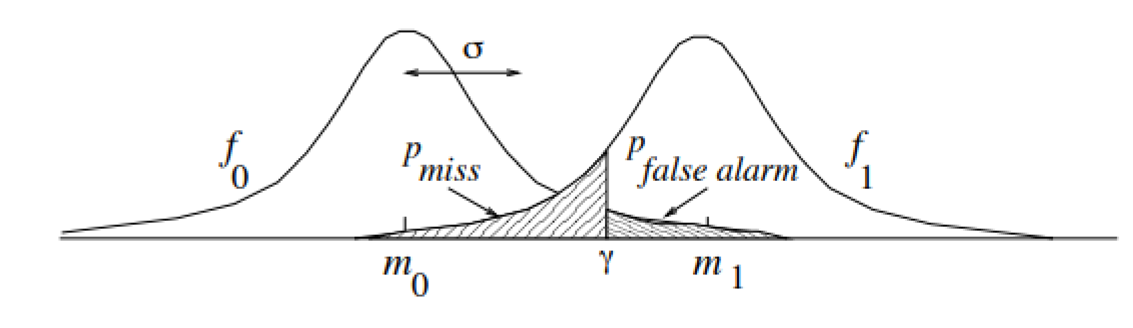
- MAP rule –

Example



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- $P_{miss} =$
- $P_{false\ alarm} =$

Slido

$f_1 = Unif([2,6]), f_0 = Unif([0,3]).$ Assume $\frac{\pi_1}{\pi_0} = \frac{3}{4}$

- Using MAP rule, the region we claim H_1

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$(a)\ None$

$(b)\ [2,3]$

$(c)\ [2,6]$

$(d)\ [3,6]$

Exam II until here

Jointly Distributed Random Variables

Motivation

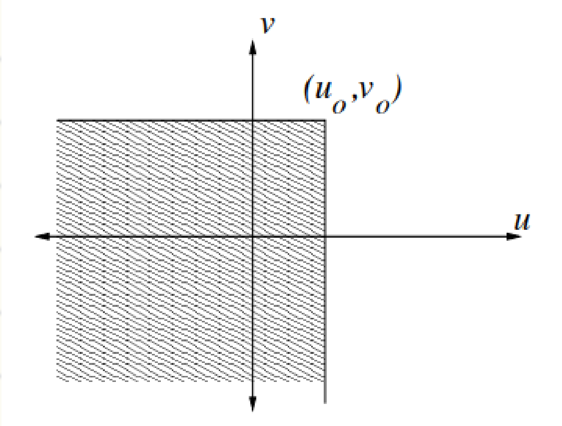
Given X and Y, we have learnt

- Independence
- Function & Scaling (e.g. X=3Y-2)

But real-world cases are more complex

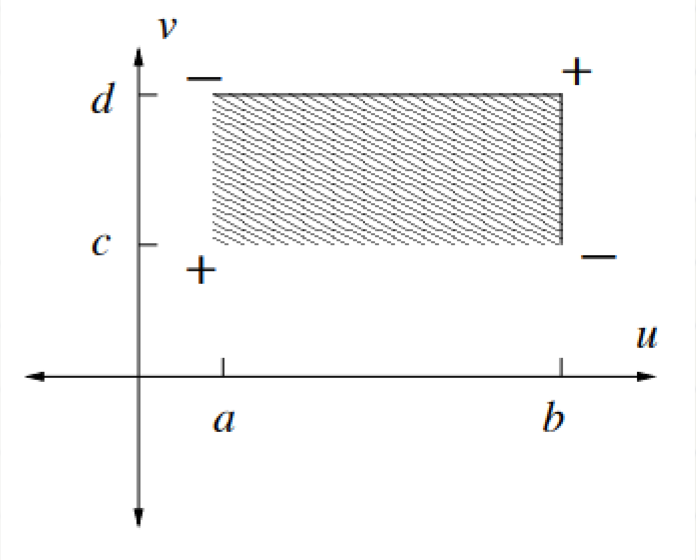
- How to show the "   " between X and Y
- "Jointly distributed" RVs
- $F_X(u) \quad \rightarrow \quad F_{X,Y}(u,v)$
- $P_X(u) \quad \rightarrow \quad P_{X,Y}(u,v)$

Joint CDF



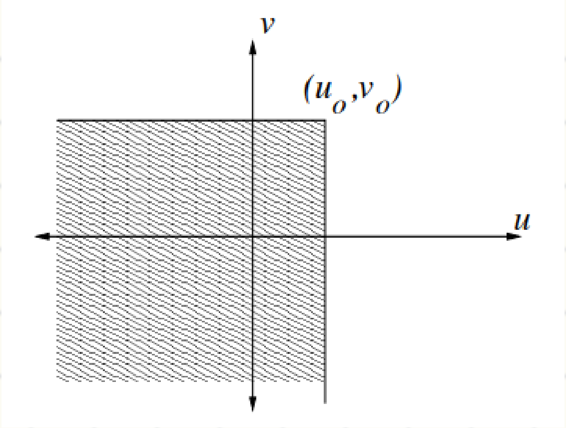
Joint CDF of X and Y

- $F_{X,Y}(u,v) = P\{X < u, Y < v\}$ for any $(u,v) \in \mathbb{R}^2$
- Completely defines all events concerning X and Y
- For a 2D rectangle region $R = (a,b] \times (c,d]$
  - $P\{(X,Y) \in R\} =$



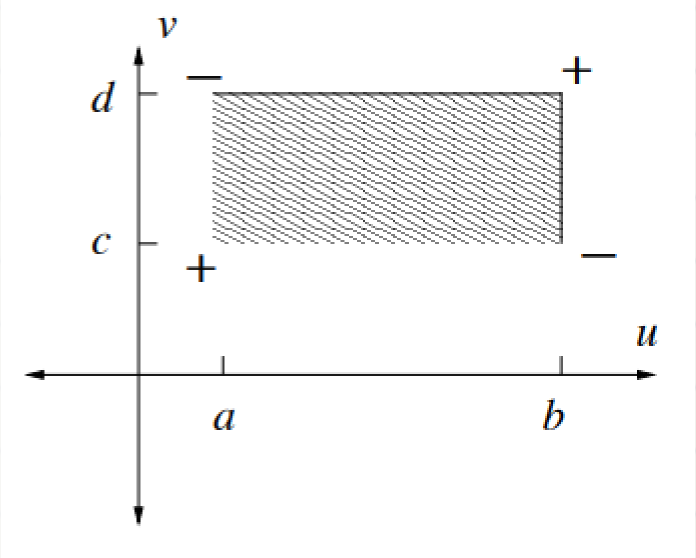
- $F_X(u) = F_{X,Y}(u,\infty)$

Joint CDF Properties



Denote $F_{X,Y}$ as F

- $0 \leq F(u,v) \leq 1$ for all $(u,v) \in \mathbb{R}^2$
- along u and along v respectively
  - F is none decreasing
  - F is right-continuous
- $\lim_{u \to -\infty} F(u,v) = 0$
- $\lim_{u \to \infty} \lim_{v \to \infty} F(u,v) = 0$



Joint PMF

If X and Y are discrete, joint PMF $p_{X,Y}(u,v) = P\{X = u, Y = v\}$

Marginalization

- Getting single RV PMF/ PDF from joint PMF/ PDF
- $p_X(u) = \sum_{v_j} p_{X,Y}(u, v_j)$ called "marginal PMF"

Conditional PMF

- $p_{Y|X}(v|u_0) = \frac{p_{X,Y}(u_0,v)}{p_X(u_0)}$

Example

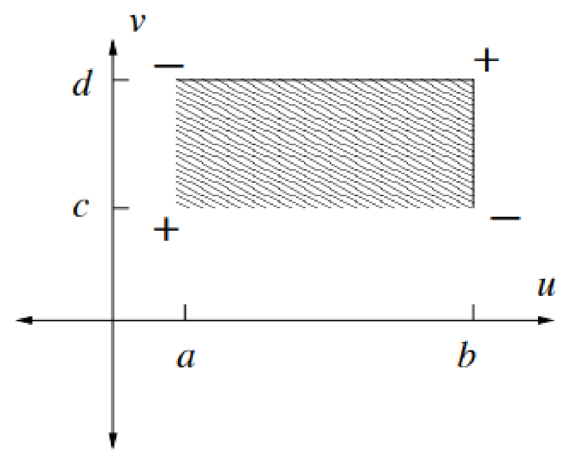
Given joint PMF $p_{X,Y}$ as the table, find

- p_X
- p_Y
- $P\{X = Y\}$
- $P\{X > Y\}$
- $p_{Y|X}(v|2)$

$Y = 3$	0.1	0.1	
$Y = 2$		0.2	0.2
$Y = 1$		0.3	0.1
	$X = 1$	$X = 2$	$X = 3$

Joint PDF

Joint PDF



If X and Y are continuous, joint PDF $f_{X,Y}(u,v)$ s.t.

- $F_{X,Y}(u,v) = \int_{-\infty}^{u}\int_{-\infty}^{v} f_{X,Y}(u,v)dvdu$
- $P\{(X,Y) \in R\} = \int\int_{R} f_{X,Y}(u,v)dudv$ for piecewise diff. R
- $f_{X,Y}(u,v) \geq 0$ for any $(u,v) \in \mathbb{R}^2$
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v)dudv =$

LOTUS: $E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v)f_{X,Y}(u,v)dudv$

- $E[X] =$

Joint PDF

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v) f_{X,Y}(u,v)\,du\,dv$$

- $E[aX + bY + c] =$

Marginalization - Projection

- $f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv$
- $E[X] = \int_{-\infty}^{\infty} u \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv\,du$

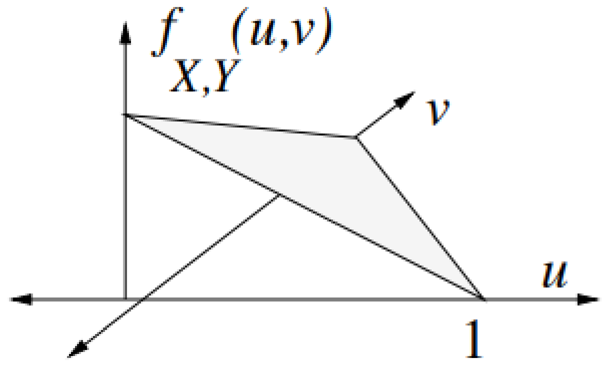
Conditional PDF - Slice

- $f_{Y|X}(v|u_0) = \frac{f_{X,Y}(u_0,v)}{f_X(u_0)}$
- Conditional expectation $E[Y|X = u_0] = \int_{-\infty}^{\infty} v f_{Y|X}(v|u_0)\,dv$

Example

Given $f_{X,Y}(u,v) = \begin{cases} c(1-u-v) & if\ u,v \geq 0, u+v \leq 1 \\ 0 & else \end{cases}$, solve

- c
- Marginal PDF $f_X(u)$
- Conditional PDF $f_{Y|X}(v|u_0)$



Uniform joint PDF

If pdf are constant over support S

- $f_{X,Y}(u,v) = \begin{cases} \frac{1}{area(S)} & if\ (u,v) \in S \\ 0 & else \end{cases}$
- S can be none-rectangular
- $P\{(X,Y) \in A\} = \frac{area(A \cap S)}{area(S)}$

Example

Let (X, Y) uniformly distributed over the uniform disk.

- $f_{X,Y}(u,v) = \begin{cases} c & if\ u^2+v^2 \leq 1 \\ 0 & else \end{cases}$, solve
- c
- $P\{X \geq 0 \cap Y \geq 0\}$
- $P\{X^2 + Y^2 \leq r^2\}$
- $f_X(u)$
- $f_{Y|X}(v|u_0)$