Last lecture

Functions of a random variable (Ch 3.8)

- General form of increasing function g
- Create uniform distribution using CDF as a function

Generating a customized RV (Ch 3.8.2)

- Intuition $g = F_X^{-1}$
- Examples
  - Uniform to Exponential
  - Uniform to D6 outcome
- Area rule – Compute $E[X]$ using F_X (Ch 3.8.3)

Agenda

Binary Hypothesis Testing with continuous distribution (Ch 3.10)

- Example

Jointly Distributed RV/ Joint CDF (Ch 4.1)

- Motivation
- Definition
- Properties

Joint PMF (Ch 4.2)

- Definition
- Example

Binary Hypothesis Testing on Continuous Distribution

Overview

Similar to discrete, but with some changes

- $P\{X = u|H_1\} \to f_1(u)$
- Likelihood Ratio $\Lambda(u) = \frac{f_1(u)}{f_0(u)}$
- LRT rule $\Lambda(X) \begin{cases} > \tau & H_1 \\ < \tau & H_0 \end{cases}$

$p_{false\ alarm}, p_{miss}, p_e$ remain the same

$\hookrightarrow P\{\text{Claim } H_1 | H_0\}$, $P\{\text{Claim } H_0 | H_1\}$

$P\{\text{Claim } H_1, H_0\} + P\{\text{Claim } H_0, H_1\}$

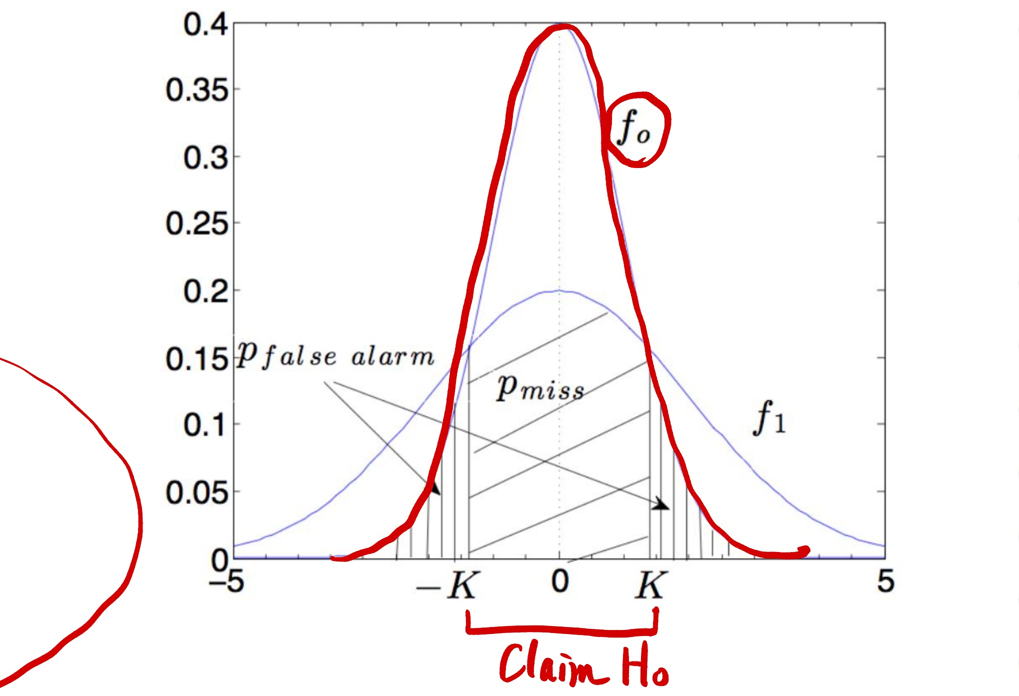
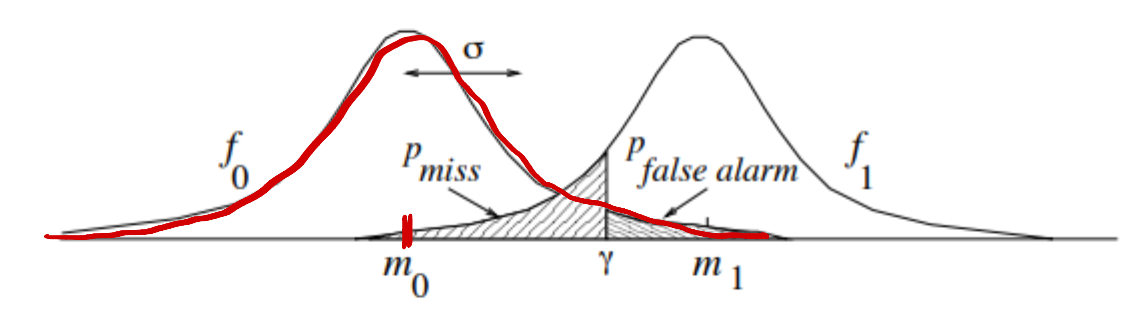


Figure 3.27: $N(0,1)$ and $N(0,4)$ pdfs and ML threshold K.

Example



X under H_i follows $N(m_i, \sigma^2)$. Given m_i, σ, π_i, Find ML and MAP rule

$\Rightarrow$ Asking $\tau_{ML} = 1$

τ_{MAP}

- $f_i(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(u-m_i)^2}{2\sigma^2}\right\}$
- $\Lambda(u) = \exp\{(u - \frac{m_0+m_1}{2})(\frac{m_1-m_0}{\sigma^2})\}$
- ML rule –

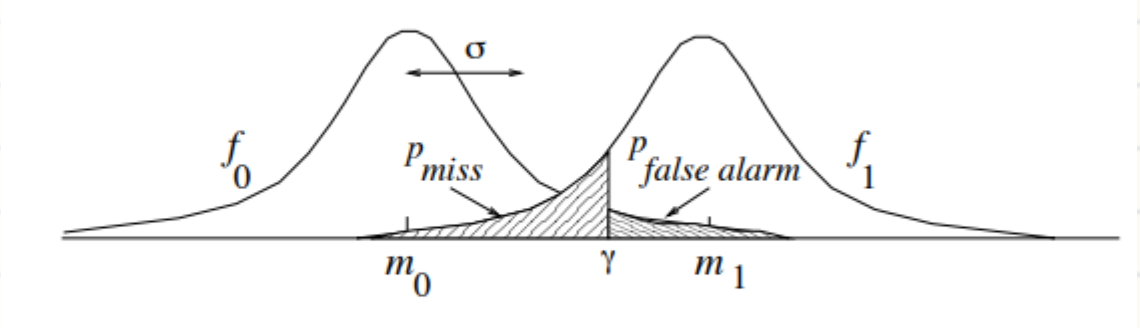
$\Lambda(u) > 1$ H_1

ML $\begin{cases} u > \boxed{\frac{m_0+m_1}{2}} & \text{Claim } H_1 \\ \text{otherwise} & \text{Claim } H_0 \end{cases}$

$\Rightarrow \gamma_{ML} \triangleq$ "decision boundary" (Not always exist)

$\Rightarrow$ Threshold on X-axis.

Example



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- $f_i(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(u-m_i)^2}{2\sigma^2}\right\}$
- $\Lambda(u) = \exp\{(u - \frac{m_0+m_1}{2})(\frac{m_1-m_0}{\sigma^2})\}$
- MAP rule –

Recall MAP $\begin{cases} \Lambda > \boxed{\frac{\pi_0}{\pi_1}} & H_1 \\ \Lambda < \frac{\pi_0}{\pi_1} & H_0 \end{cases}$ τ_{MAP}

$\downarrow \ln$

$\begin{cases} \ln \Lambda > \ln \tau_{MAP} \\ \ln \Lambda < \ln \tau_{MAP} \end{cases}$

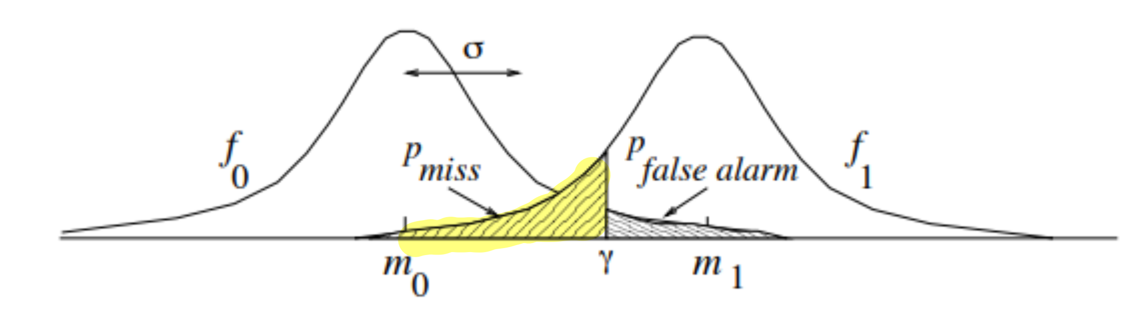
$$\left(\mu - \frac{m_0+m_1}{2}\right)\left(\frac{m_1-m_0}{\sigma^2}\right) \gtrless \ln \tau_{MAP}.$$

$$\Rightarrow \mu \gtrless \left(\frac{\sigma^2}{m_1-m_0}\right)\ln\left(\frac{\pi_0}{\pi_1}\right) + \frac{m_1+m_0}{2}.$$

τ_{MAP}

$\gamma_{MAP}.$

Example



X under H_i follows $N(m_i, \sigma^2)$. Given m_i, σ, π_i, Find ML and MAP rule

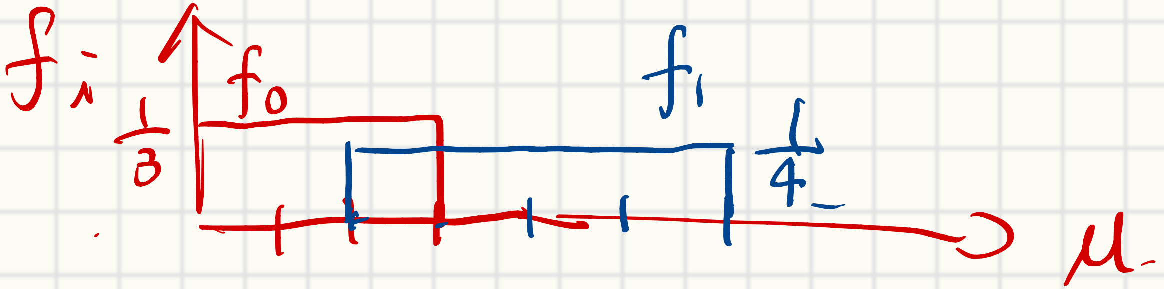
- $f_i(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(u-m_i)^2}{2\sigma^2}\right\}$
- $\Lambda(u) = \exp\{(u - \frac{m_0+m_1}{2})(\frac{m_1-m_0}{\sigma^2})\}$
- $P_{miss} =$ $P(X < \gamma \mid H_1) = P\left(\underbrace{\boxed{\frac{X-m_1}{\sigma}}}_{}\, < \frac{\gamma - m_1}{\sigma} \,\middle|\, H_1\right)$ Z.

  $= Q\left(\frac{m_1-\gamma}{\sigma}\right) = \Phi\left(\frac{\gamma-m_1}{\sigma}\right)$

- $P_{false\ alarm} =$

  $Q\left(\frac{\gamma-m_0}{\sigma}\right)$

Slido



$f_1 = Unif([2,6]),\, f_0 = Unif([0,3])$. Assume $\frac{\pi_1}{\pi_0} = \frac{3}{4}$

- Using MAP rule, the region we claim H_1

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(a) $None$

(b) $[2,3]$

(c) $[2,6]$

(d) $[3,6]$

$$\Lambda(\mu) = \frac{f_1}{f_0} = \begin{cases} 0 & \mu \in [0,2] \\ \frac{3}{4}, & \mu \in [2,3] \\ \infty, & \mu \in [3,6] \end{cases}$$

$$\Lambda > \frac{\pi_0}{\pi_1} = \frac{4}{3}$$

Exam II until here

Jointly Distributed Random Variables

Motivation

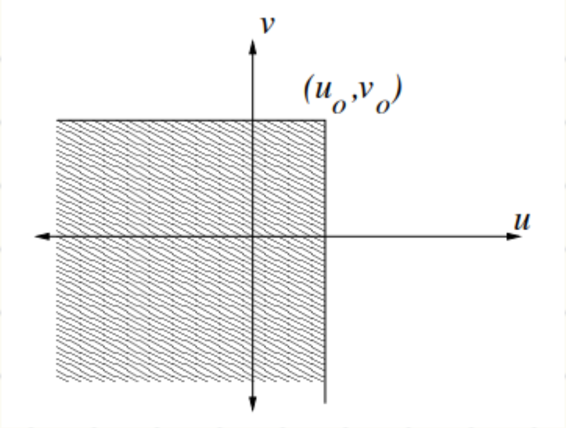
Given X and Y, we have learnt

- Independence
- Function & Scaling (e.g. X=3Y-2)

But real-world cases are more complex

- How to show the "   " between X and Y
- "Jointly distributed" RVs
- $F_X(u) \quad \rightarrow \quad F_{X,Y}(u, v)$
- $P_X(u) \quad \rightarrow \quad P_{X,Y}(u, v)$

Joint CDF

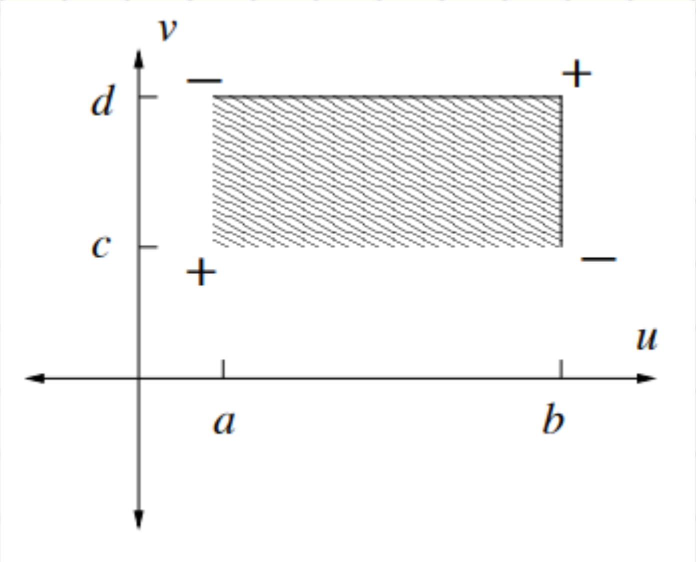


AND

Joint CDF of X and Y

- $F_{X,Y}(u,v) = P\{X \le u, Y \le v\}$ for any $(u,v) \in \mathbb{R}^2$
- Completely defines all events concerning X and Y
- For a 2D rectangle region $R = (a,b] \times (c,d]$
  - $P\{(X,Y) \in R\} =$ $F_{XY}(b,d) - F_{XY}(a,d) - F_{XY}(b,c) + F_{XY}(a,b)$

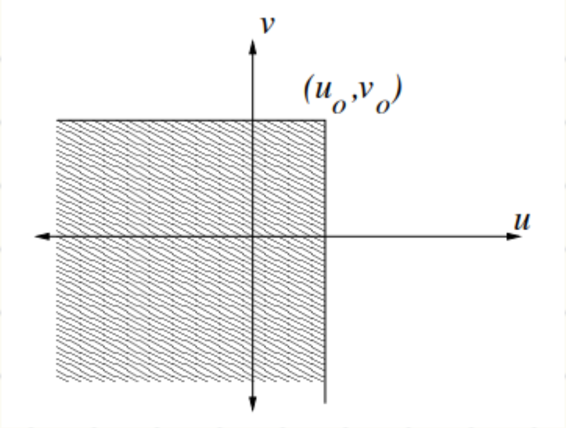
$a \le X \le b \quad c \le Y \le d$



- $F_X(u) = F_{X,Y}(u,\infty)$

$F_Y(v) = F_{X,Y}(\infty, v)$

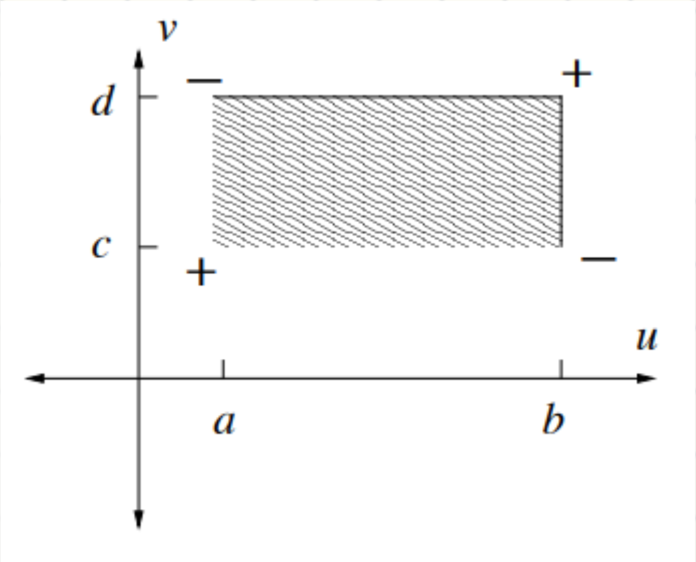
Joint CDF Properties



Denote $F_{X,Y}$ as F

- $0 \leq F(u,v) \leq 1$ for all $(u,v) \in \mathbb{R}^2$
- along u and along v respectively
  - F is none decreasing
  - F is right-continuous
- $\lim\limits_{u \to -\infty} F(u,v) = 0$
- $\lim\limits_{u \to \infty} \lim\limits_{v \to \infty} F(u,v) = \cancel{0}\ 1$

$\lim\limits_{u \to \infty} F(u,v) = F_{XY}(\infty, v) = F_Y(v)$



Joint PMF

Roll 2 D6 $\begin{cases} X \\ Y \end{cases}$ $P_{X,Y}(3,4) = \frac{1}{36}$

If X and Y are discrete, joint PMF $p_{X,Y}(u,v) = P\{X = u, Y = v\}$

Marginalization

- Getting single RV PMF/ PDF from joint PMF/ PDF
- $p_X(u) = \underline{\sum_{v_j} p_{X,Y}(u, v_j)}$ called "marginal PMF"

$Y = v_j$ are partitions

Conditional PMF

- $p_{Y|X}(v|u_0) = \frac{p_{X,Y}(u_0,v)}{p_X(u_0)}$

Example

Given joint PMF $p_{X,Y}$ as the table, find

0.2	$Y=3$	0.1	0.1	
0.4	$Y=2$		0.2	0.2
0.4	$Y=1$		0.3	0.1
		$X=1$	$X=2$	$X=3$
		0.1	0.6	0.3

0.2

- p_X $\quad P_X(u) = \sum_{v} P_{XY}(u,v)$

$P_X(u) = [0.1, 0.6, 0.3]$

- p_Y

$P_Y(v) = [0.4, 0.4, 0.2]$

- $P\{X = Y\}$ $= \sum_{u} P_{XY}(u,u) = 0.2$
- $P\{X > Y\}$ $= \sum_{u} \sum_{v<u} P_{XY}(u,v) = 0.3 + 0.1 + 0.2 = 0.6$
- $p_{Y|X}(v|2)$

$= P\{Y=v \mid X=2\} = \dfrac{P_{XY}(v,2)}{P_X(2)} = \dfrac{[0.3, 0.2, 0.1]}{0.6}$

$$P_{Y|X}(v|2) = \left[\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right]$$

Joint PDF