Last lecture

Functions of a random variable (Ch 3.8)

- Find CDF/ PDF of g(X) (Ch 3.8.1)
  - Examples for
    - General
    - X is Gaussian
    - Case by case
    - g is cosine/ tangent
    - g is strictly increasing
- Generating random variables with $F_X(c)$ (Ch 3.8.2)

Agenda

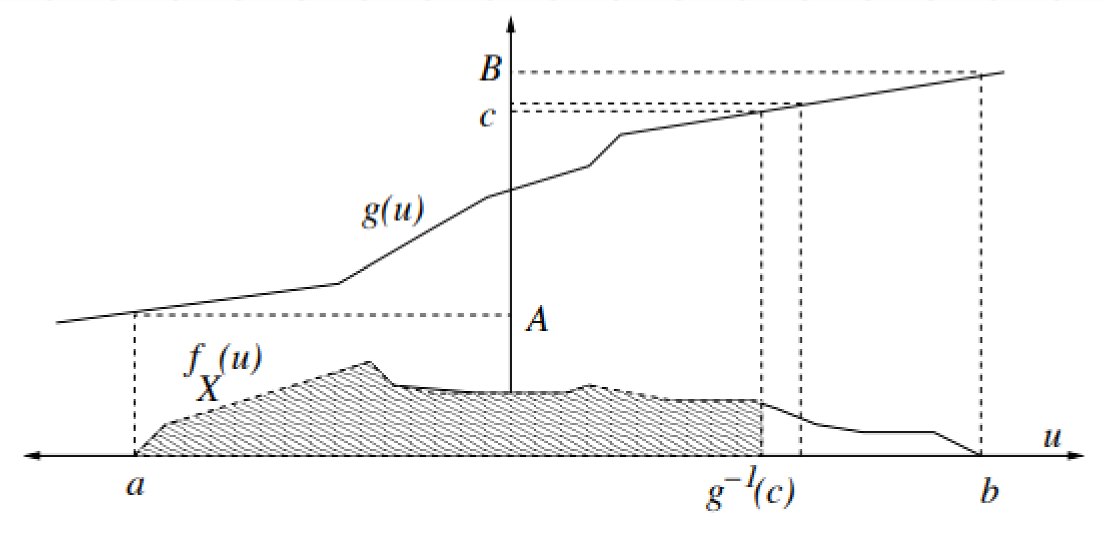
Functions of a random variable (Ch 3.8)

- General form of increasing function g
- Create uniform distribution using CDF as a function

Generating a customized RV (Ch 3.8.2)

- Intuition $g = F_X^{-1}$
- Examples
  - Uniform to Exponential
  - Uniform to D6 outcome
- Area rule – Compute $E[X]$ using F_X (Ch 3.8.3)

Increasing g function



- X has support $[a, b]$
- $Y = g(X)$
- g is strictly increasing
- Y has support $[g(a), g(b)] = [A, B]$
- Find $F_Y(c)$ where $A \leq c \leq B$

- $f_Y(c) =$

Generating a random variable

Creating uniform distribution

- Let $Y = F_X(X)$
- What is F_Y?

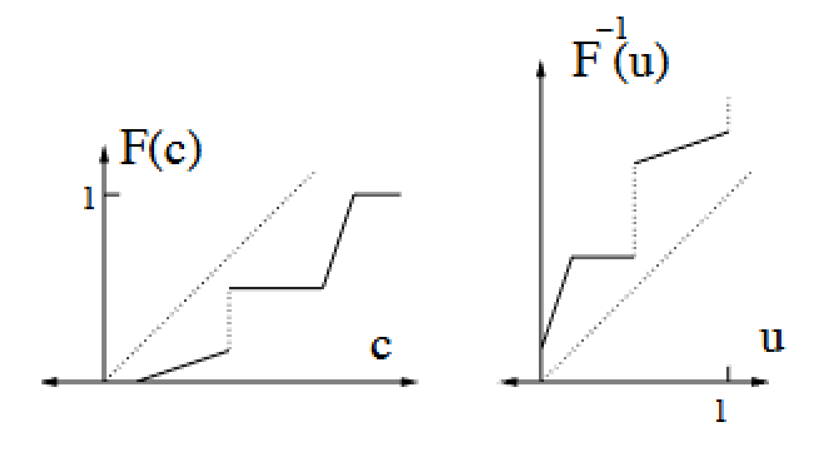
Generating a customized RV

If we want to generate (sample) $X = g(U \sim Uni([0,1]))$ to fit F_X

- $U = F_X(X)$
- $F_X^{-1}(U) = X$
- $g = F_X^{-1}$, but F_X^{-1} may not be a function
  - $F_X(c_1) = F_X(c_2)$
- For any F, define

$$F^{-1}(u) = \min\{c: F(c) \geq u\}$$

- $F: c \to u, \qquad F^{-1}: u \to c$



$F^{-1}(u_0) \leq c_0$ if and only if $u_0 \leq F(c_0)$

- $F_X(c) = P\{F^{-1}(U) \leq c\} = P\{U \leq F(c)\} = F(c)$

Example – Uniform to Exponential

Find g s.t. $g(U) \sim Exp(\lambda = 1)$ when $U \sim Uniform([0,1])$

- $F(c) = 1 - e^{-c} = u$ for $c \geq 0$
- Find $g(u) = F^{-1}(u)$ in terms of u

- $1 - e^{-c} = u$
- $c = -\ln(1 - u)$
- $g(u) = F^{-1}(u) = -\ln(1 - u)$ for $0 \leq u \leq 1$
- Check, if $c \geq 0$
  - $P\{-\ln(1 - U) \leq c\} = P\{\ln(1 - U) \geq -c\}$
    $= P\{1 - U \geq e^{-c}\}$
    $= P\{U \leq 1 - e^{-c}\} = F(c)$

Example – Uniform to Exponential

Find g s.t. $g(U) \sim Exp(\lambda = 1)$ when $U \sim Uniform([0,1])$

- $F(c) = 1 - e^{-c} = u$ for $c \geq 0$
- Find $g(u) = F^{-1}(u)$ in terms of u

- $1 - e^{-c} = u$
- $c = -\ln(1 - u)$
- $g(u) = F^{-1}(u) = -\ln(1 - u)$ for $0 \leq u \leq 1$
- Note - g is not unique
  - E.g. U and $1 - U$ are all uniform, so $-\ln u$ is also a valid g

Example – Uniform to Dice outcome

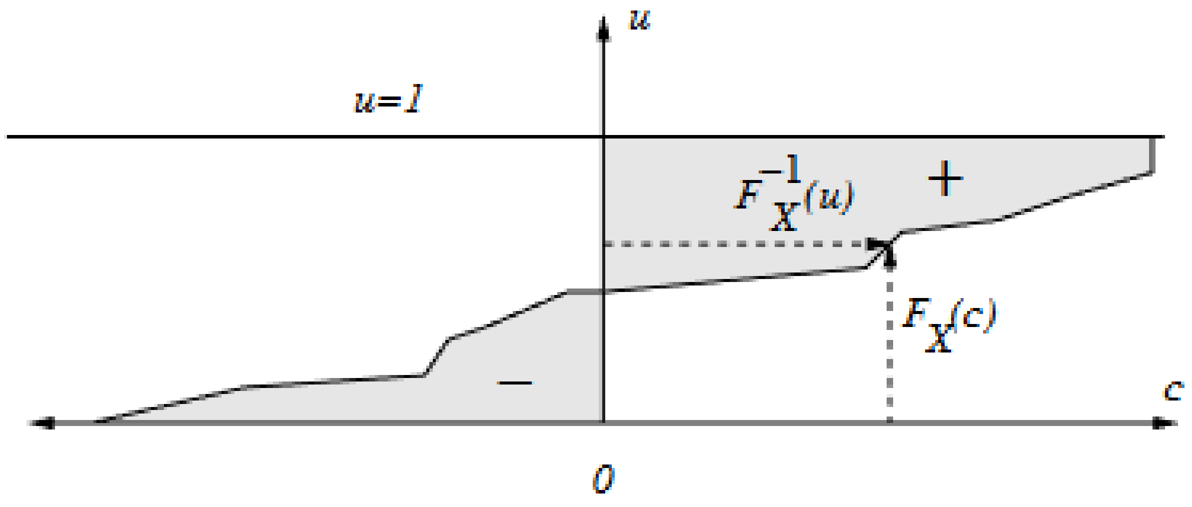
Find g s.t. $g(U) \sim \{$"Fair die distribution"$\}$ when $U \sim Uniform([0,1])$

- $F(i) = \frac{\lfloor i \rfloor}{6} = u$ for $1 \leq i \leq 6$
- $g(u) = F^{-1}(u)$
- $g(u) = i$ where $\frac{i-1}{6} \leq u \leq \frac{i}{6}$ for $0 \leq u \leq 1$

The area rule for expectation based on the CDF

$$E[X] = Area^{+} - Area^{-} = \int_0^{\infty}\left(1 - F_X(c)\right)dc - \int_{-\infty}^{0} F_X(c)dc$$

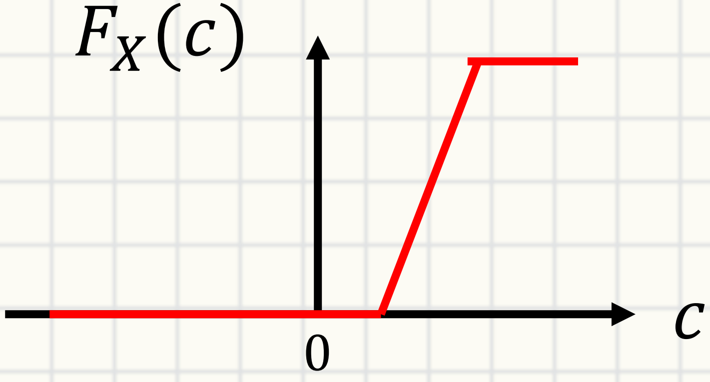
$$E[X] = E[F_X^{-1}(U)] = \int_0^1 F_X^{-1}(u)du$$



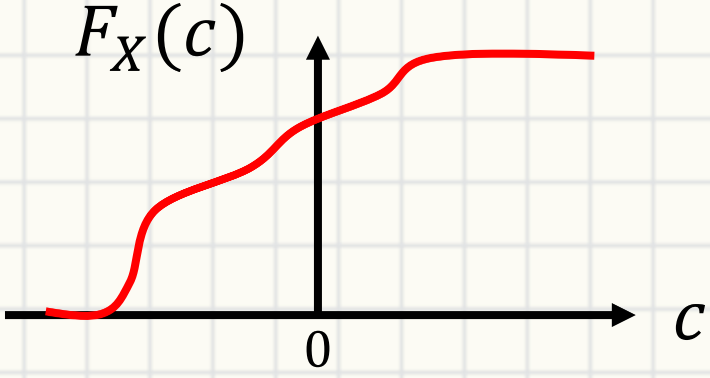
Slido

Order the $E[X]$ from high to low

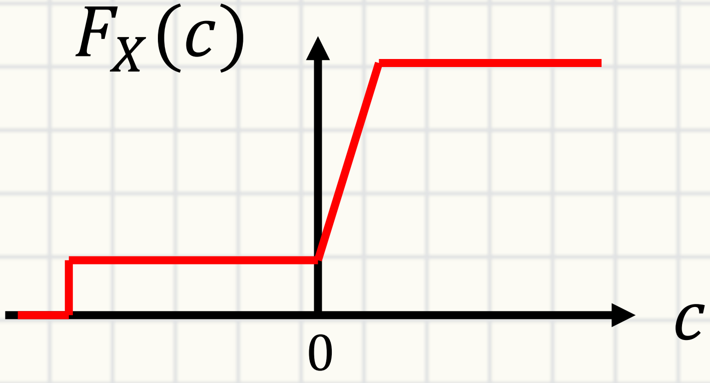
(A)



(B)



(C)



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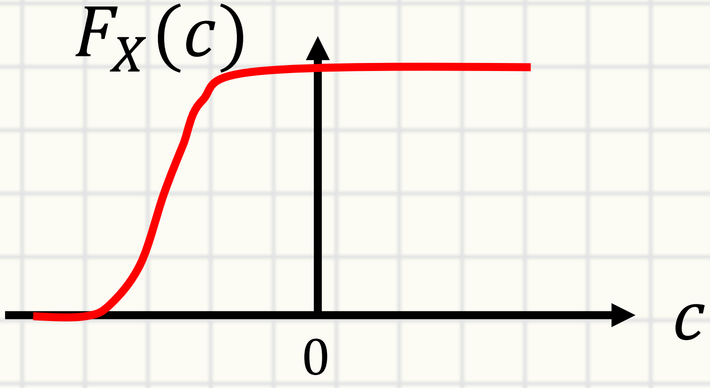


Binary Hypothesis Testing on Continuous Distribution

Overview

Similar to discrete, but with some changes

- $P\{X = u|H_1\} \rightarrow f_1(u)$
- Likelihood Ratio $\Lambda(u) =$
- LRT rule $\Lambda(X) =$

$p_{false\ alarm}, p_{miss}, p_e$ remain the same

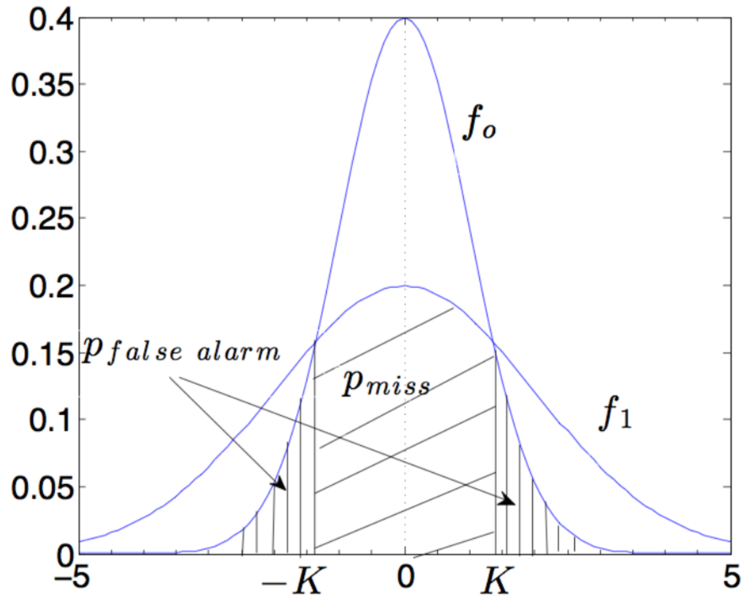
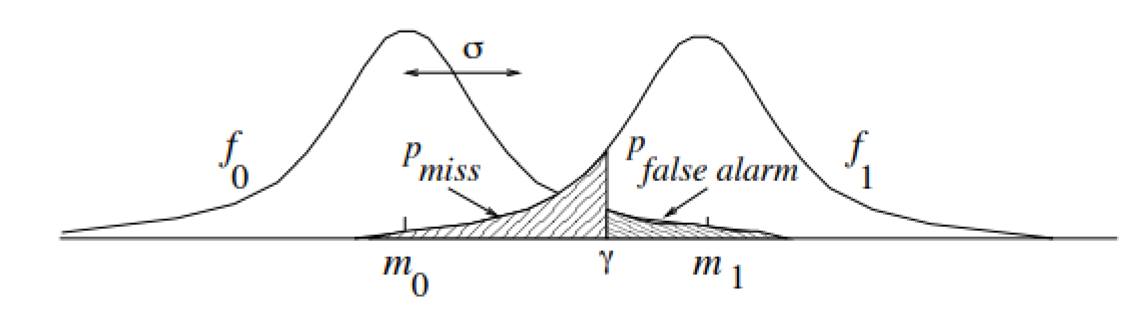


Figure 3.27: $N(0,1)$ and $N(0,4)$ pdfs and ML threshold K.

Example



X under H_i follows $N(m_i, \sigma^2)$. Given m_i, σ, π_i, Find ML and MAP rule

- $f_i(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(u-m_i)^2}{2\sigma^2}\}$
- $\Lambda(u) = \exp\{(u - \frac{m_0+m_1}{2})(\frac{m_1-m_0}{\sigma^2})\}$
- ML rule –
- MAP rule –