Last lecture

Functions of a random variable (Ch 3.8)

- Find CDF/ PDF of g(X) (Ch 3.8.1)
  - Examples for
    - General
    - X is Gaussian
    - Case by case
    - g is cosine/ tangent
    - g is strictly increasing
- Generating random variables with $F_X(c)$ (Ch 3.8.2)

Agenda

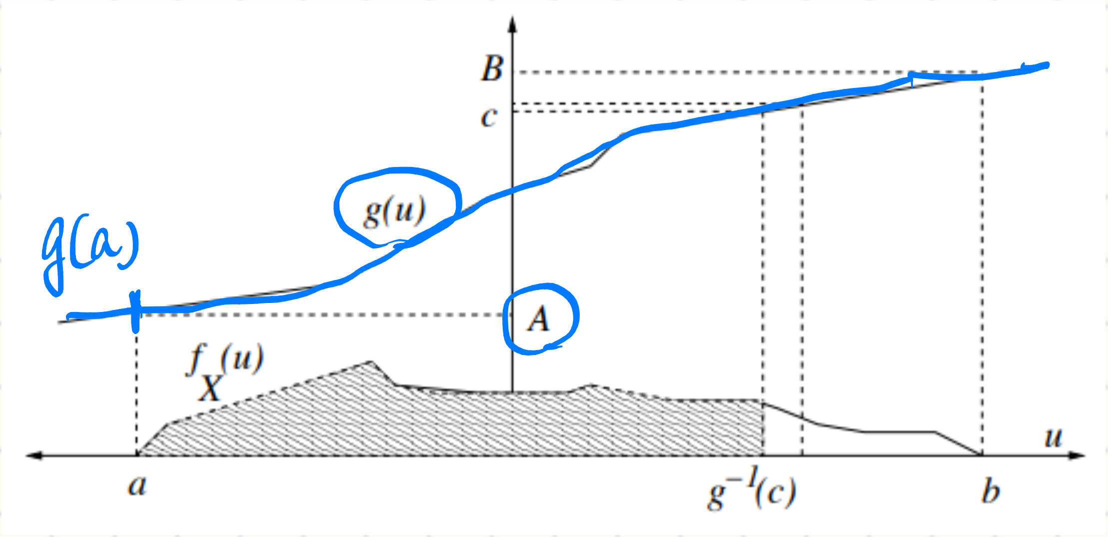
Functions of a random variable (Ch 3.8)

- General form of increasing function g
- Create uniform distribution using CDF as a function

Generating a customized RV (Ch 3.8.2)

- Intuition $g = F_X^{-1}$
- Examples
  - Uniform to Exponential
  - Uniform to D6 outcome
- Area rule – Compute $E[X]$ using F_X (Ch 3.8.3)

Increasing g function



- X has support $[a, b]$
- $Y = g(X)$
- g is strictly increasing
- Y has support $[g(a), g(b)] = [A, B]$
- Find $F_Y(c)$ where $A \le c \le B$

$g(x) = c$.

$F_Y(c) = P\{X \le g^{-1}(c)\} = F_X(g^{-1}(c))$

- $f_Y(c) =$

Example, $Y = aX + b \quad a > 0$.

$F_Y(c) = F_X\left(\frac{c-b}{a}\right)$

$F_X'(g^{-1}(c)) = f_X(g^{-1}(c)) \cdot \frac{1}{g'(g^{-1}(c))}$

$$g(g^{-1}(c)) = c$$

$$\frac{\partial}{\partial c} \downarrow$$

$$g'(g^{-1}(c)) \frac{d g^{-1}(c)}{d c} = 1$$

Generating a random variable

Creating uniform distribution

- Let $Y = F_X(X)$
- What is F_Y?

$Y = g(X)$ Let $g(\cdot) \triangleq F_X(\cdot)$

$Y \in [0, 1]$

$Y \sim X$

$F_X(Y)$

$= P\{X \leq Y\}$

$F_Y(c) = P\{F_X(X) \leq c\} = c$

$Y \sim \text{Uniform}[0, 1]$

$\Rightarrow$ For any RV X, $F_X(X) \sim \text{Uniform}[0, 1]$

Generating a customized RV

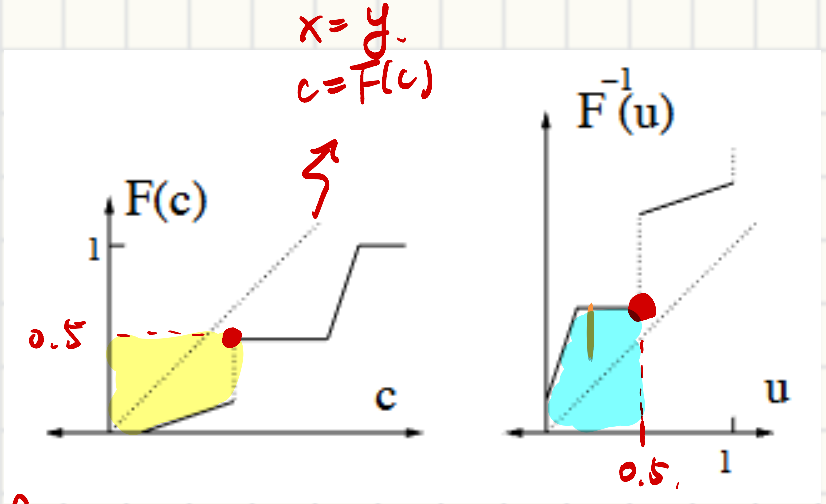
If we want to generate (sample) $X = g(U \sim Uni([0,1]))$ to fit F_X

- $U = F_X(X)$
- $F_X^{-1}(U) = X$ *core for sampling.*
- $g = F_X^{-1}$, but F_X^{-1} may not be a function
  - $F_X(c_1) = F_X(c_2)$ *Multiple to 1.*
- For any F, define

$$F^{-1}(u) = \min\{c : F(c) \geq u\}$$

left most of F(c) or bottom of $F^{-1}(u)$

- $F: c \to u, \qquad F^{-1}: u \to c$



$F^{-1}(u_0) \leq c_0$ if and only if $u_0 \leq F(c_0)$

- $F_X(c) = P\{F^{-1}(U) \leq c\} = P\{U \leq F(c)\} = F(c)$

Example – Uniform to Exponential

$T \sim Exp(\lambda)$

$F_T(t) = 1 - e^{-\lambda t}$

Find g s.t. $g(U) \sim Exp(\lambda = 1)$ when $U \sim Uniform([0,1])$

- $F(c) = 1 - e^{-c} = u$ for $c \geq 0$
- Find $g(u) = F^{-1}(u)$ in terms of u

$F(c) = u \quad F^{-1}(u) = c$

- $1 - e^{-c} = u$
- $c = -\ln(1-u)$ ⇒ Write c in terms of u.
- $g(u) = F^{-1}(u) = -\ln(1-u)$ for $0 \leq u \leq 1$

↳ One valid $F^{-1}(c)$

- Check, if $c \geq 0$ (support of $Exp(\lambda)$)
  - $P\{-\ln(1-U) \leq c\}$ $= P\{\ln(1-U) \geq -c\}$
    $= P\{1 - U \geq e^{-c}\}$
    $= P\{U \leq 1 - e^{-c}\} = F(c)$

Example – Uniform to Exponential

Find g s.t. $g(U) \sim Exp(\lambda = 1)$ when $U \sim Uniform([0,1])$

- $F(c) = 1 - e^{-c} = u$ for $c \geq 0$
- Find $g(u) = F^{-1}(u)$ in terms of u

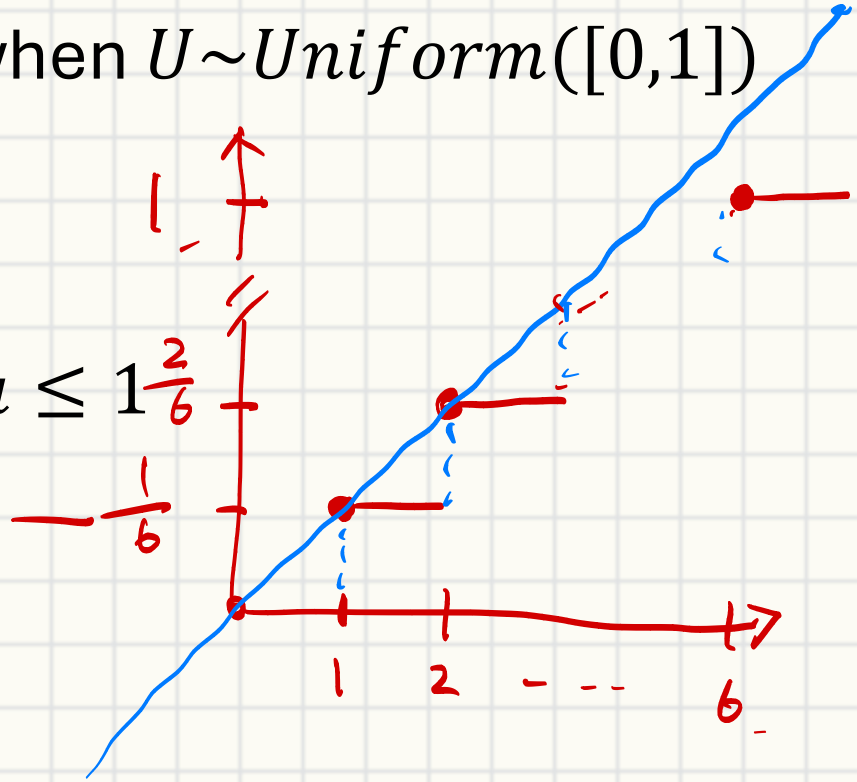
- $1 - e^{-c} = u$
- $c = -\ln(1 - u)$
- $g(u) = F^{-1}(u) = -\ln(1 - u)$ for $0 \leq u \leq 1$
- Note - g is not unique
  - E.g. U and $1 - U$ are all uniform, so $-\ln u$ is also a valid g

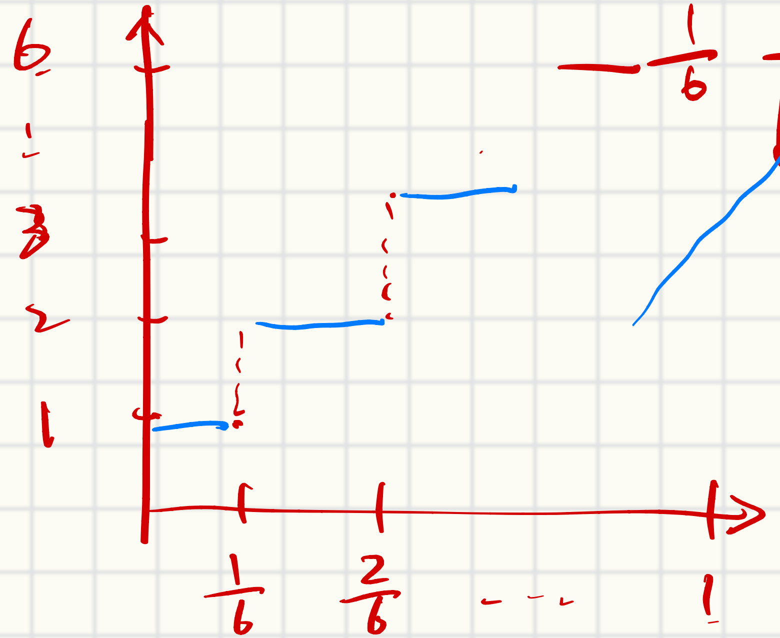
$1-u \to u.$

Example – Uniform to Dice outcome

Find g s.t. $g(U) \sim \{$“Fair die distribution”$\}$ when $U \sim Uniform([0,1])$

- $F(i) = \frac{\lfloor i \rfloor}{6} = u$ for $1 \le i \le 6$
- $g(u) = F^{-1}(u)$
- $g(u) = i$ where $\frac{i-1}{6} \le u \le \frac{i}{6}$ for $0 \le u \le 1$



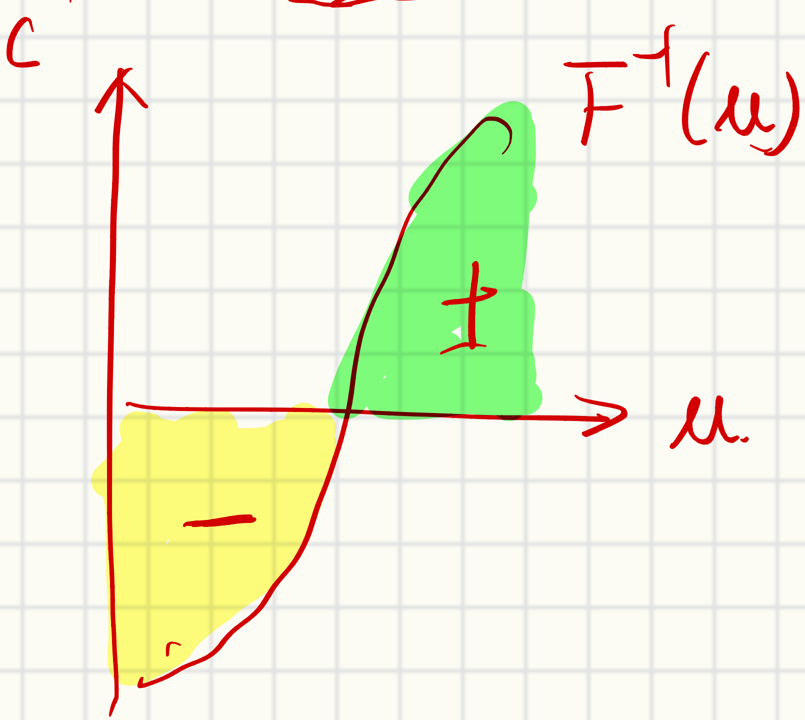


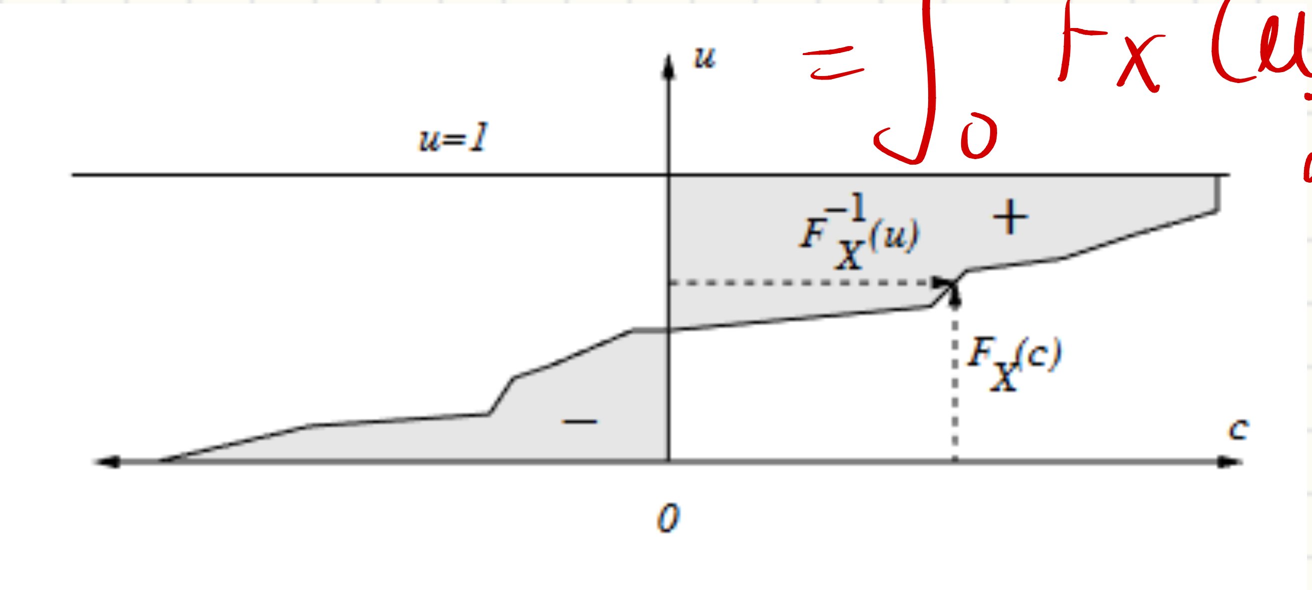
The area rule for expectation based on the CDF

$$E[X] = Area^{+} - Area^{-} = \int_0^{\infty}(1 - F_X(c))dc - \int_{-\infty}^{0} F_X(c)dc$$

$$E[X] = E[F_X^{-1}(U)] = \int_0^1 F_X^{-1}(u)du$$

$$E[Y] = \int_{-\infty}^{\infty} y f_U(y) dy = \int_0^1 F_X^{-1}(u)\, du$$

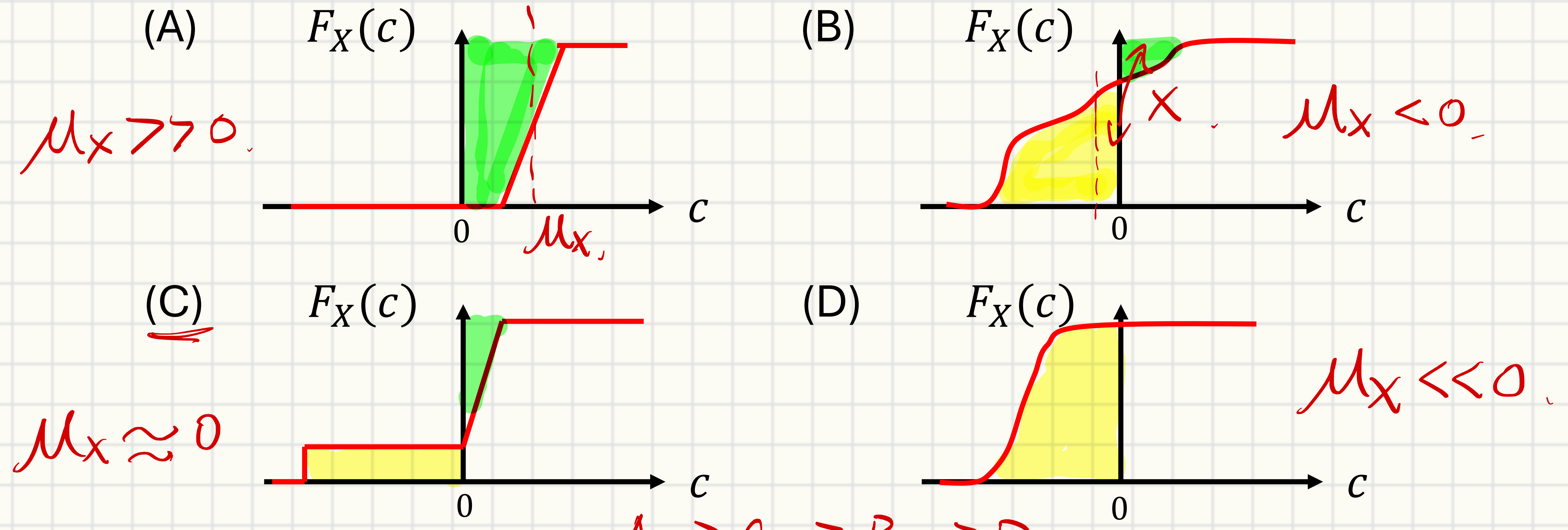




Slido

#3033794

Order the $E[X]$ from high to low



A → C → B → D

Binary Hypothesis Testing on Continuous Distribution

Overview

Similar to discrete, but with some changes

- $P\{X = u|H_1\} \rightarrow f_1(u)$ => likelihood function.
- Likelihood Ratio $\Lambda(u) = \frac{f_1(u)}{f_0(u)}$
- LRT rule $\Lambda(X) = \begin{cases} > \tau & H_1 \\ < \tau & H_0 \end{cases}$

$p_{false\ alarm}, p_{miss}, p_e$ remain the same

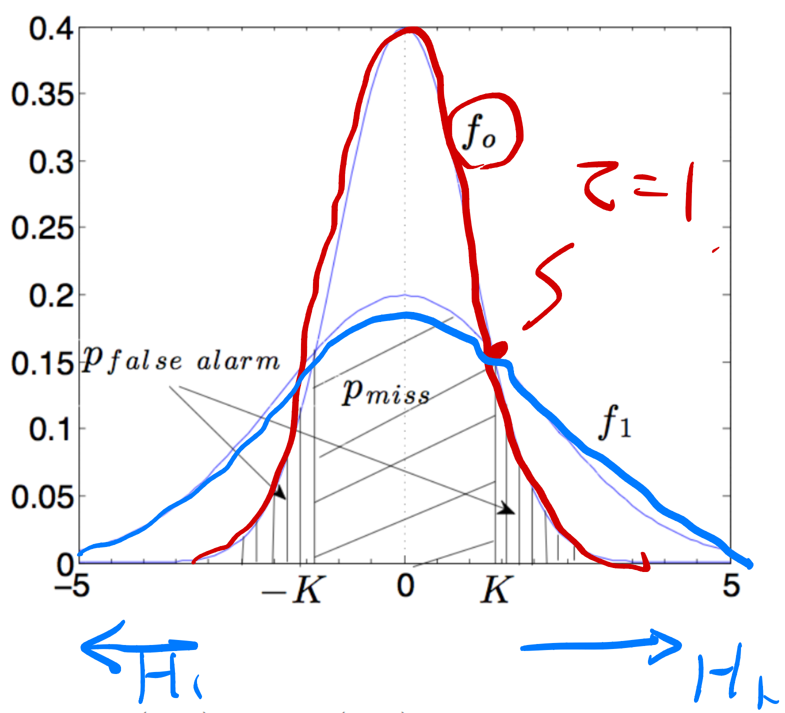


Figure 3.27: $N(0,1)$ and $N(0,4)$ pdfs and ML threshold K.