Last lecture

Gaussian (normal) Distribution (Ch 3.6.2)

- Example

The Central Limit Theorem and Gaussian Approximation (Ch 3.6.3)

- Definition
- CDF Approximation
- Examples

Agenda

Functions of a random variable (Ch 3.8)

- Find CDF/ PDF of g(X) (Ch 3.8.1)
  - Examples for
    - General
    - X is Gaussian
    - Case by case
    - g is cosine/ tangent
    - g is strictly increasing
- Generating random variables with $F_X(c)$ (Ch 3.8.2)

Examples

Suppose $\mu_X = 10$ and $\sigma_X^2 = 3$. Compute $P\{X < 10 - \sqrt{3}\}$ if

- X is a Gaussian RV in terms of Q
- X is a uniform RV

(Hint: $10 - \sqrt{3} \approx 8.27$)

Example

We want to estimate p with $\hat{p} = \frac{X}{n}$

- Find $P\{|\hat{p} - p| < \delta\}$ in terms of n, p, δ, and Φ

- Find δ w/ 99% confidence if $p = 0.5$, $n = 1000$. Given that $\Phi(2.58) \approx 0.995$

- What if $p = 0.1$?

Functions of a random variable

Find CDF/ PDF of g(X)

Motivation – I know X follows some distribution

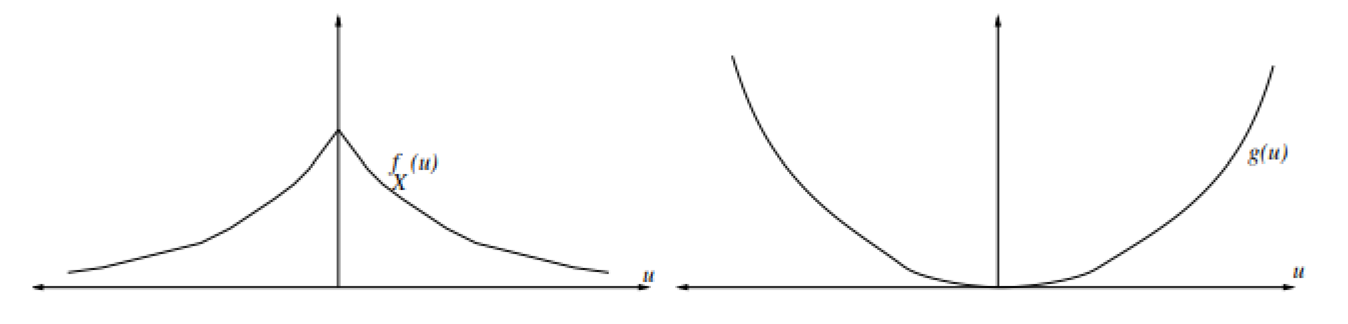
- but what about $Y = g(X)$?

1. Scope the problem - Find support of X and Y, are they continuous or discrete?
2. Find $F_Y(c)$ from integrating $f_X(x)$ over $\{x: g(x) \leq c\}$
   - If Y is discrete, normally we can find pmf $p_Y(c)$
3. Get $f_Y = F_Y{}'$

Examples - General

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\le c} f_X(x)dx$
3. $f_Y = F_Y{}'$

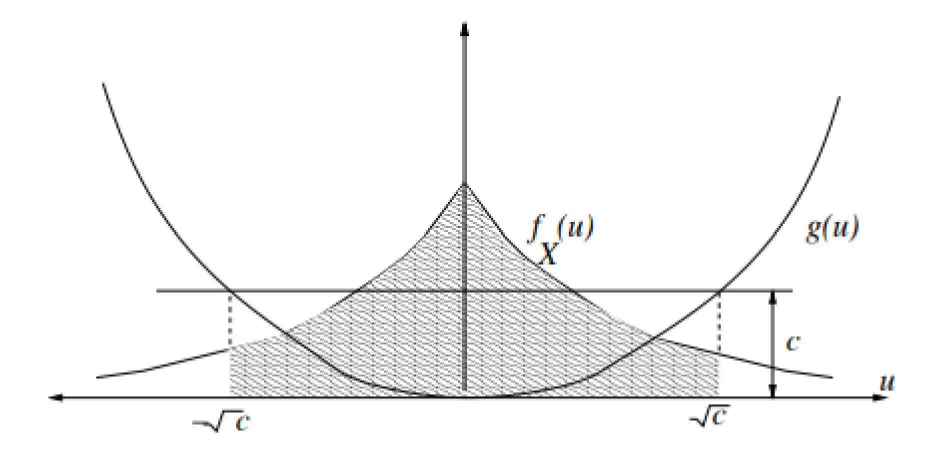
RV X follows $f_X(u) = \frac{e^{-|u|}}{2}$ for $u \in R$. $Y = X^2$. Find f_Y, μ_Y and σ_Y^2



1. $F_Y =$

2. $P\{X^2 \le c\} =$

3. $f_Y(c) =$



Examples - Gaussian

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\leq c} f_X(x)dx$
3. $f_Y = F_Y{}'$

Let $Y = X^2$, where $X \sim N(\mu = 2, \sigma^2 = 3)$, find f_Y

- $F_Y(c) =$

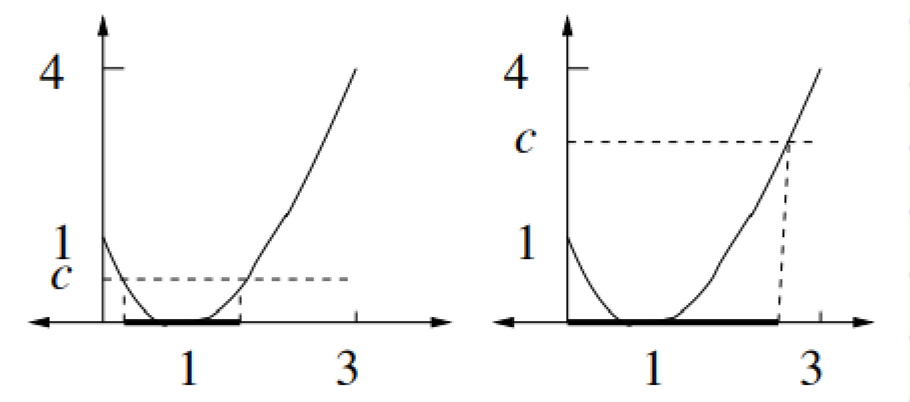
- $f_Y(c) = F_Y{}'(c)$

Examples – Case by case

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\leq c} f_X(x)dx$
3. $f_Y = F_Y{}'$

Let $Y = (X-1)^2$, where $X \sim Uni(0,3)$, find f_Y

- $F_Y(c)$
  - $0 \leq c \leq 1$
  - $1 \leq c \leq 4$
- $f_Y(c) = F_Y{}'(c)$

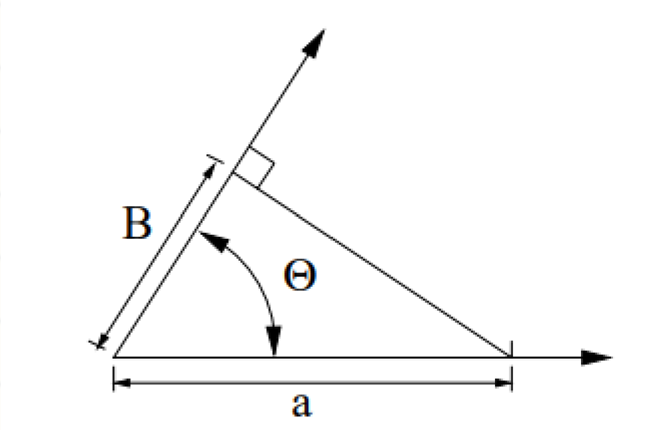


Examples – Cosine

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\leq c} f_X(x)dx$
3. $f_Y = F_Y{}'$

Let $B = acos\Theta$, where $\Theta \sim Uni(-\pi, \pi)$, find f_B

- Useful in DSP
- $F_B(c)$
- $f_B(c) = F_B{}'(c)$
  - Hint $\frac{d(\cos^{-1} x)}{dx} = -(1-x)^{\frac{1}{2}}$



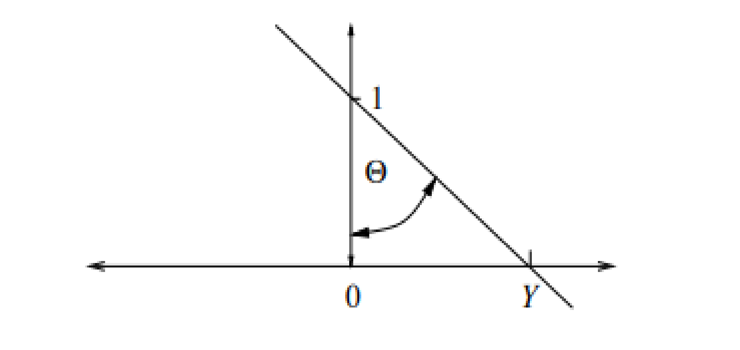
Examples – Tangent

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\leq c} f_X(x)dx$
3. $f_Y = F_Y{}'$

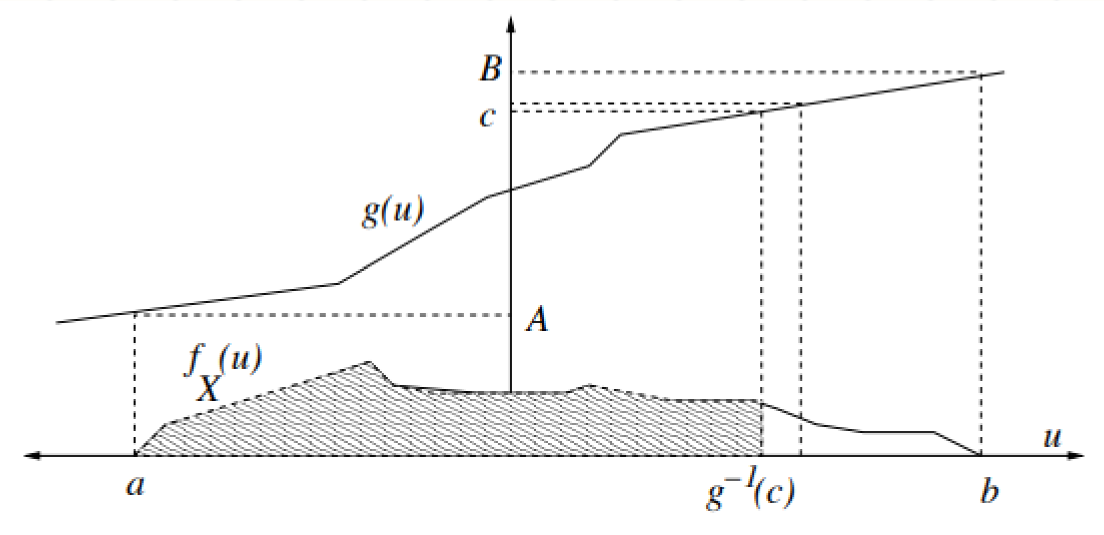
Let $Y = \tan\Theta$, where $\Theta \sim Uni\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, find f_Y

- $F_Y(c)$

- $f_Y(c) = F_Y{}'(c)$



Increasing g function



- X has support $[a, b]$
- $Y = g(X)$
- g is strictly increasing
- Y has support $[g(a), g(b)] = [A, B]$
- Find $F_Y(c)$ where $A \leq c \leq B$

- $f_Y(c) =$

Generating a random variable

Creating uniform distribution

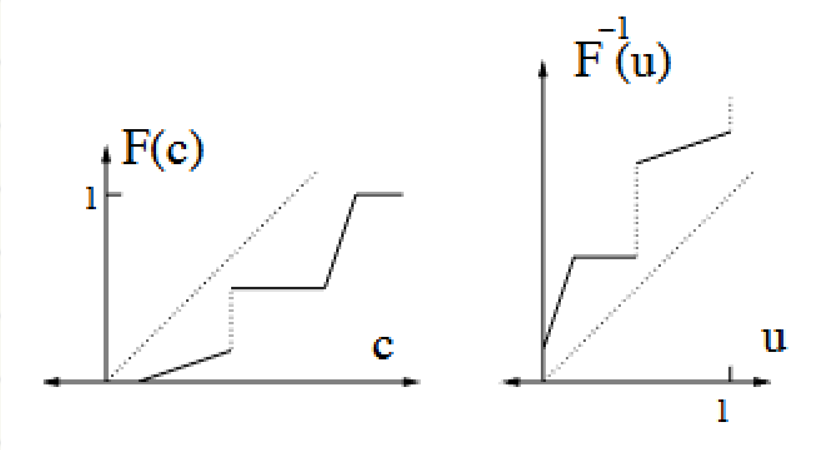
- Let $Y = F_X(X)$
- What is F_Y?

Generating a customized RV

If we want to generate (sample) $X = g(U \sim Uni([0,1]))$ to fit F_X

- $U = F_X(X)$
- $F_X^{-1}(U) = X$
- $g = F_X^{-1}$, but F_X^{-1} may not be a function
  - $F_X(c_1) = F_X(c_2)$
- For any F, define

$$F^{-1}(u) = \min\{c: F(c) \geq u\}$$



$F^{-1}(u_0) \leq c_0$ if and only if $u_0 \leq F(c_0)$

- $F_X(c) = P\{F^{-1}(U) \leq c\} = P\{U \leq F(c)\} = F(c)$

Example – Uniform to Exponential

Find g s.t. $g(U) \sim Exp(\lambda = 1)$ when $U \sim Uniform([0,1])$

- $F(c) = 1 - e^{-c} = u$ for $c \geq 0$
- Find $g(u) = F^{-1}(u) = c$ in terms of u

- $1 - e^{-c} = u$
- $c = -\ln(1 - u)$
- $g(u) = F^{-1}(u) = -\ln(1 - u)$ for $0 \leq u \leq 1$
- Check, if $c \geq 0$
  - $P\{-\ln(1 - U) \leq c\} = P\{\ln(1 - U) \geq -c\}$
    $= P\{1 - U \geq e^{-c}\}$
    $= P\{U \leq 1 - e^{-c}\} = F(c)$

Example – Uniform to Exponential

Find g s.t. $g(U) \sim Exp(\lambda = 1)$ when $U \sim Uniform([0,1])$

- $F(c) = 1 - e^{-c} = u$ for $c \geq 0$
- Find $g(u) = F^{-1}(u)$ in terms of u

- $1 - e^{-c} = u$
- $c = -\ln(1 - u)$
- $g(u) = F^{-1}(u) = -\ln(1 - u)$ for $0 \leq u \leq 1$
- Note - g is not unique
  - E.g. U and $1 - U$ are all uniform, so $-\ln u$ is also a valid g

Example – Uniform to Dice outcome

Find g s.t. $g(U) \sim \{$"Fair die distribution"$\}$ when $U \sim Uniform([0,1])$

- $F(i) = \frac{\lfloor i \rfloor}{6} = u$ for $1 \leq i \leq 6$
- $g(u) = F^{-1}(u)$
- $g(u) = i$ where $\frac{i-1}{6} \leq u \leq \frac{i}{6}$ for $0 \leq u \leq 1$

Puzzle – Generate a fair coin toss

Exam Safe

How can we generate a fair coin toss result from a biased coin?

- Assume we do not know p

The area rule for expectation based on the CDF

$$E[X] = Area^{+} - Area^{-} = \int_{0}^{\infty} \left(1 - F_X(c)\right) dc - \int_{-\infty}^{0} F_X(c) dc$$

$$E[X] = \int_{0}^{1} F_X^{-1}(u) du$$