Last lecture

Gaussian (normal) Distribution (Ch 3.6.2)

- Example

The Central Limit Theorem and Gaussian Approximation (Ch 3.6.3)

- Definition
- CDF Approximation
- Examples

Agenda

Functions of a random variable (Ch 3.8)

- Find CDF/ PDF of g(X) (Ch 3.8.1)
  - Examples for
    - General
    - X is Gaussian
    - Case by case
    - g is cosine/ tangent
    - g is strictly increasing
- Generating random variables with $F_X(c)$ (Ch 3.8.2)

Example

$\frac{X-\mu_X}{\sigma_X}$ $\quad \frac{X-np}{\sqrt{np(1-p)}}$ $\quad P\{\|Z\| \le \delta c\} = \Phi(\delta c) - \Phi(-\delta c)$

$= 2\Phi(\delta c) - 1$

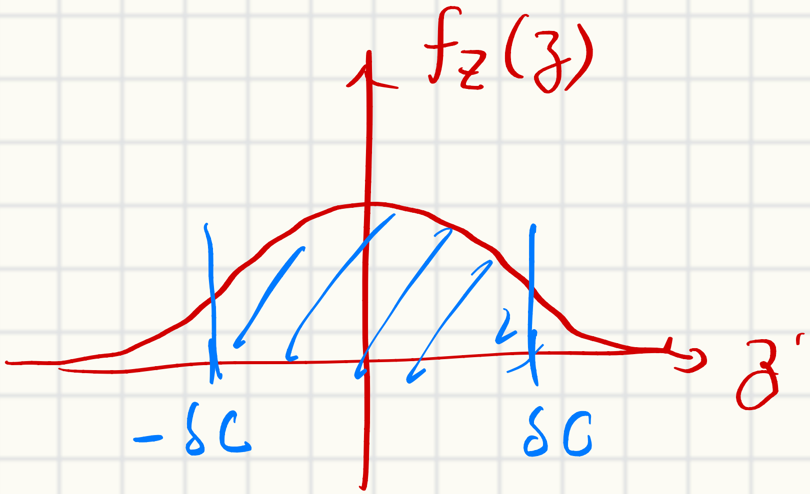
We want to estimate p with $\hat{p} = \frac{X}{n}$

- Find $P\{|\hat{p} - p| < \delta\}$ in terms of n, p, δ, and Φ

$P\left\{\left|\frac{X-np}{n}\right| < \delta\right\}$ $\left(\times \boxed{\frac{\sqrt{n}}{\sqrt{p(1-p)}}}\ \text{both}\right)$ $\Rightarrow P\left\{\left|\frac{X-np}{\sqrt{np(1-p)}}\right| \le \delta c\right\}$ $\quad c$

- Find δ w/ 99% confidence if $p = 0.5$, $n = 1000$. Given that $\Phi(2.58) \approx 0.995$

- What if $p = 0.1$?



$$P\{|\hat{p}-p|<\delta\} = 0.99 \approx 2\Phi(\delta c)-1$$

$$\Phi(\delta c) = \frac{1.99}{2} = 0.995 = \Phi(2.58)$$

$$\delta\sqrt{\frac{n}{p(1-p)}} = 2.58 \qquad n=1000, \quad p=0.5$$

$$\Rightarrow \delta = 0.04$$

$$\delta\sqrt{\frac{n}{p(1-p)}} = 2.58 \qquad n=1000 \quad p=0.1$$

$$\Rightarrow \delta = 0.025$$

Functions of a random variable

Find CDF/ PDF of g(X)

$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{\|a\|}$

Motivation – I know X follows some distribution

- but what about $Y = g(X)$? if $Y = g(X) = aX + b$

$g(x) = x^2, \|x\|, \cos x$ ...

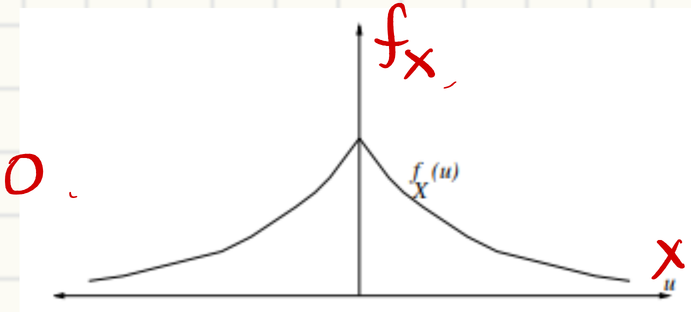
1. Scope the problem - Find support of X and Y, are they continuous or discrete? LOTUS
2. Find $F_Y(c)$ from integrating $f_X(x)$ over $\{x: g(x) \le c\}$
   - If Y is discrete, normally we can find pmf $p_Y(c)$
3. Get $f_Y = F_Y{}'$

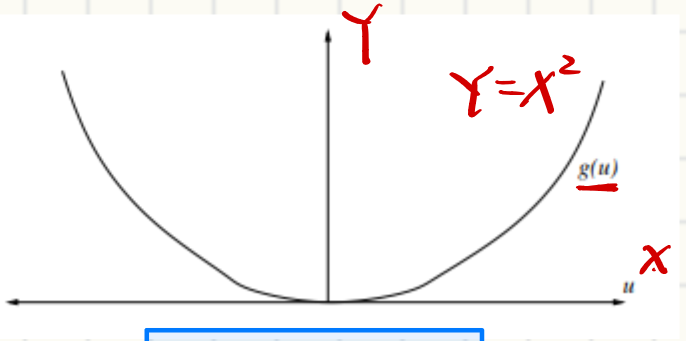
Examples - General

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\le c} f_X(x)dx$
3. $f_Y = F_Y{}'$

RV X follows $f_X(u) = \dfrac{e^{-|u|}}{2}$ for $u \in R$. $Y = X^2$. Find f_Y, μ_Y and σ_Y^2

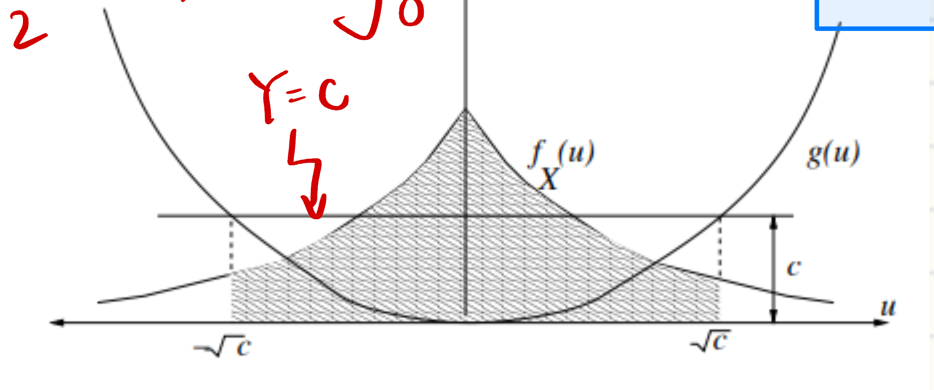
1. $F_Y(c) = \begin{cases} 0 & c \le 0 \\ \boxed{1 - e^{-c^{\frac{1}{2}}}} & c > 0 \end{cases}$





2. $\underbrace{P\{X^2 \le c\}}$ $=$

$$\int_{-\sqrt{c}}^{\sqrt{c}} f_X(u)du = \int_{-\sqrt{c}}^{\sqrt{c}} \frac{e^{-|u|}}{2} du = \int_0^{\sqrt{c}} e^{-u} du = \boxed{1 - e^{-\sqrt{c}}}$$



3. $f_Y(c) = \begin{cases} 0 & c \le 0 \\ \dfrac{e^{-\sqrt{c}}}{2\sqrt{c}} & c > 0 \end{cases}$

Examples - Gaussian

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\le c} f_X(x)dx$
3. $f_Y = F_Y{}'$

Let $Y = X^2$, where $X \sim N(\mu = 2, \sigma^2 = 3)$, find f_Y

$P\{0 \le Y \le 0+\epsilon\} \overset{?}{=} 0$

- $F_Y(c) = \begin{cases} 0 & c<0 \\ ? & c \ge 0 \end{cases}$
- $f_Y(c) = F_Y{}'(c)$

$\longrightarrow P\{-\sqrt{c} \le X \le \sqrt{c}\}$

$P\left\{\frac{-\sqrt{c}-2}{\sqrt{3}} \le \frac{X-2}{\sqrt{3}} \le \boxed{\frac{\sqrt{c}-2}{\sqrt{3}}}\right\}$

$= \Phi\left(\frac{\sqrt{c}-2}{\sqrt{3}}\right) - \Phi\left(\frac{-\sqrt{c}-2}{\sqrt{3}}\right)$

$$\Phi'(s) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-s^2}{2}\right)$$

$$f_Y = \frac{1}{\sqrt{24\pi c}} \left\{ \exp\left(\frac{-(\sqrt{c}-2)^2}{6}\right) + \exp\left(\frac{-(\sqrt{c}+2)^2}{6}\right) \right\}$$

Examples – Case by case

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\le c} f_X(x)dx$
3. $f_Y = F_Y'$

Let $Y = (X-1)^2$, where $X \sim Uni(0,3)$, find f_Y

- $F_Y(c)$
  - $0 \le c \le 1$

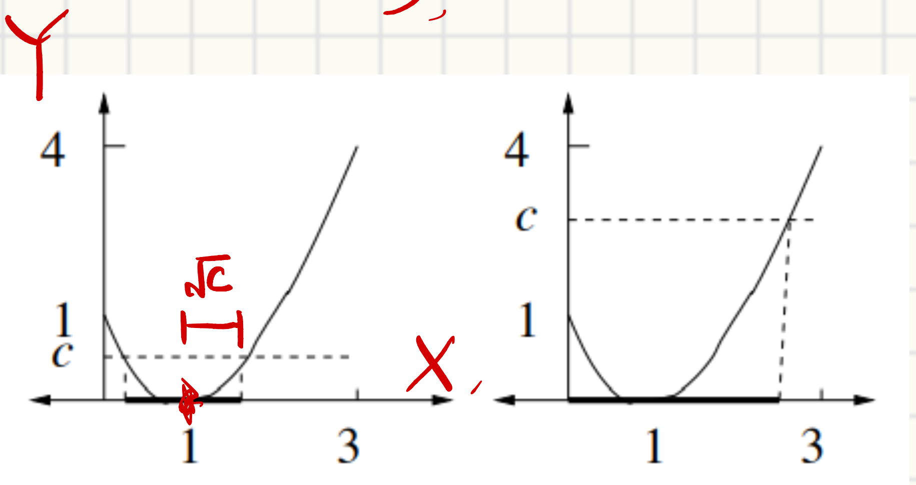
$$F_Y(c) = P\{1-\sqrt{c} \le X \le 1+\sqrt{c}\} = \int_{1-\sqrt{c}}^{1+\sqrt{c}} f_X(u)du = \frac{2\sqrt{c}}{3}$$

  - $1 \le c \le 4$

$$P\{0 \le X \le 1+\sqrt{c}\} = \frac{1+\sqrt{c}}{3}$$

- $f_Y(c) = F_Y'(c)$ 0 otherwise

$$f_Y(c) = \begin{cases} \frac{1}{3\sqrt{c}} & 0 < c \le 1 \\ \frac{1}{6\sqrt{c}} & 1 < c < 4 \end{cases}$$



Examples – Cosine

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\le c} f_X(x)dx$
3. $f_Y = F_Y{}'$

Let $B = a\cos\Theta$, where $\Theta \sim Uni(-\pi,\pi)$, find f_B

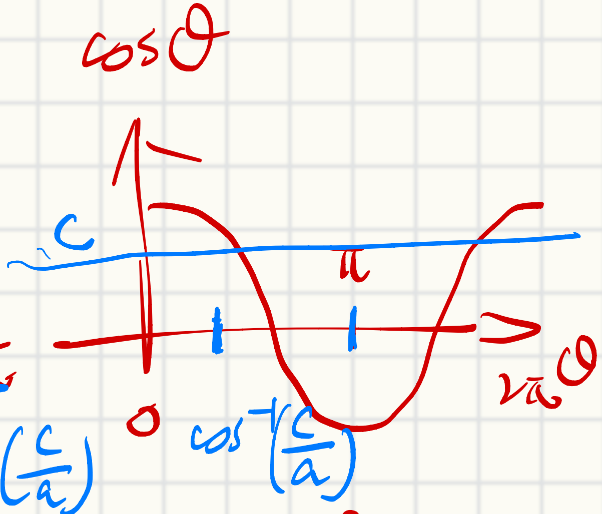
- Useful in DSP
- $F_B(c) = P\{a\cos\theta \le c\} = P\{\theta \ge \cos^{-1}(\frac{c}{a})\}$

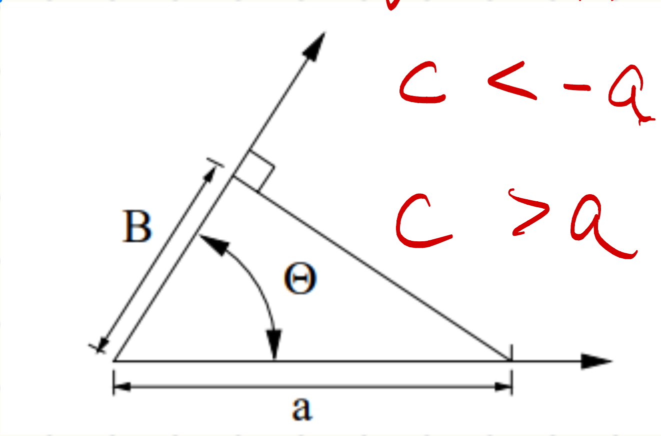
  $0 < \theta < \pi$

  $$\begin{cases} = 1 - \frac{\cos^{-1}(\frac{c}{a})}{\pi} = \frac{\pi - \cos^{-1}(\frac{c}{a})}{2\pi} \times 2 & \text{if } \|c\| \le a \\ 0 & c < -a \\ 1 & c > a \end{cases}$$

- $f_B(c) = F_B{}'(c)$
  - Hint $\frac{d(\cos^{-1} x)}{dx} = -(1-x)^{\frac{1}{2}}$

$$\frac{d\left(1 - \frac{\cos^{-1}(\frac{c}{a})}{\pi}\right)}{dc} = \frac{1}{\pi\sqrt{a^2 - c^2}}$$





Examples – Tangent

1. Find support and continuity
2. $F_Y(c) = \int_{x:f(x)\le c} f_X(x)dx$
3. $f_Y = F_Y{}'$

Let $Y = \tan\Theta$, where $\Theta \sim Uni\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, find f_Y

- $F_Y(c) = P\{\Theta \le \tan^{-1}(c)\}$

  $= \dfrac{\tan^{-1}(c) - (-\frac{\pi}{2})}{\pi}$

- $f_Y(c) = F_Y{}'(c)$

  $= \dfrac{1}{\pi\sqrt{1+c^2}}$