Last lecture

Uniform Distribution (Ch 3.3)

Exponential Distribution (Ch 3.4)
- Memoryless property
- Connect to Geo(p)

Poisson process (Ch 3.5)
- Motivation
- Bernoulli process to Poisson process
- Definition
- Properties

Agenda

Erlang Distribution (Ch 3.5.3)

Linear Scaling (Ch 3.6.1)
- Equation and derivation
- Examples

Gaussian (normal) Distribution (Ch 3.6.2)
- Motivation and Definition
- Examples

Erlang Distribution

Definition

Let T_r denotes the time of rth count of a Poisson process

- $T_r = \sum_{i=1}^{r} U_i$, $U_i \sim Exp(\lambda)$
- $F_{T_r}^c(\text{t}) = \text{P}\{\text{T}_r > \text{t}\}$: "At most $r - 1$ count by time t"
- $F_{T_r}^c(\text{t}) =$

- $f_{T_r}(t) = -\dfrac{dF_{T_r}^c(t)}{dt} =$

- Intuitively – Exactly   arrival before time t, and the rth event comes at t

Linear Scaling

Definition

Let $Y = aX + b$, where X, Y are RV and a, b are constants

- $f_Y(u) = f_X\left(\frac{u-b}{a}\right) \times \frac{1}{a}$
- Example - $X \sim Uniform(0,1)$

- $F_Y(u) = P\{aX + b < u\}$
- $f_Y(u) = F_Y'(u) =$
- $E[Y] =$

- $Var(Y) =$ $\qquad$ $\sigma_Y =$

Example

Consider the temperature in Champaign
- X denotes the temperature in C (Celcius)
- Y denotes it in F (Fahrenheit)
- $Y = (1.8)X + 32$

- Express f_Y in terms of f_X
- Find f_Y if $X \sim Uniform(15, 20)$

Example

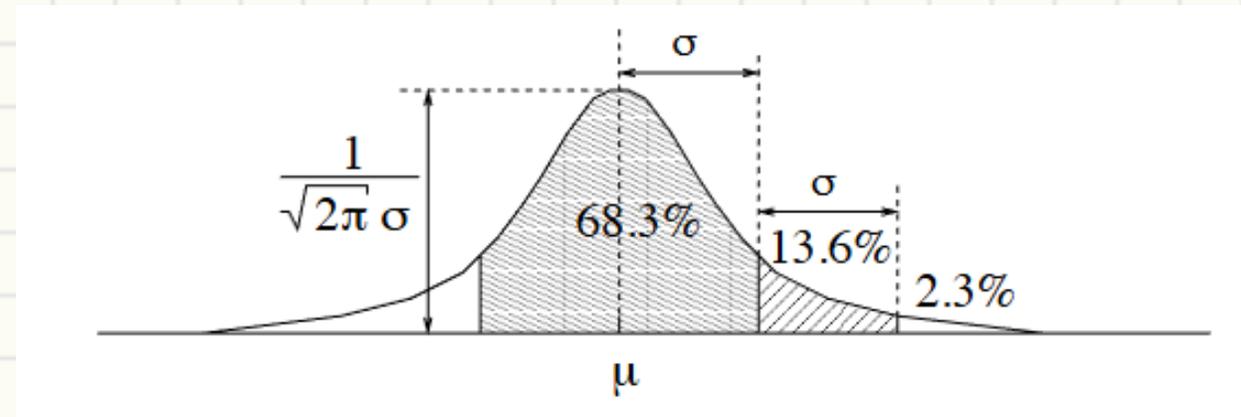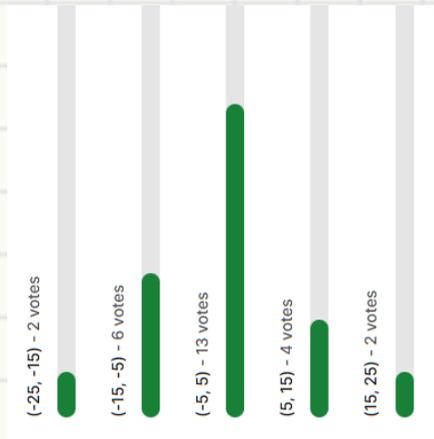Let $X \sim Uniform(a, b)$. Find the pdf of standardized RV $\frac{X - \mu_X}{\sigma_X}$

- Recall the variance of $Uniform(0,1)$ is 12

Gaussian (Normal) Distribution

Definition

A normal distribution is defined by μ_X and σ_X^2, Let $X \sim N(\mu, \sigma^2)$

- $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$

- Usage – model "Sum of many small independent events"
- E.g. Sum of many binomial distributions

Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

$$X \sim N(\mu = 0, \sigma^2 = 1) \sim N(0,1)$$

- $\Phi(u) \triangleq F_X(u) =$

- $Q(u) = 1 - \Phi(u)$

- Pre-computed tables!

Φ and Q tables

z	−0.00	−0.01	−0.02	−0.03	−0.04	−0.05	−0.06	−0.07	−0.08	−0.09
-3.9	0.00005	0.00005	0.00004	0.00004	0.00004	0.00004	0.00004	0.00004	0.00003	0.00003
-3.8	0.00007	0.00007	0.00007	0.00006	0.00006	0.00006	0.00006	0.00005	0.00005	0.00005
-3.7	0.00011	0.00010	0.00010	0.00010	0.00009	0.00009	0.00008	0.00008	0.00008	0.00008

⋮

z	−0.00	−0.01	−0.02	−0.03	−0.04	−0.05	−0.06	−0.07	−0.08	−0.09
−0.1	0.46017	0.45620	0.45224	0.44828	0.44433	0.44038	0.43644	0.43251	0.42858	0.42465
−0.0	0.50000	0.49601	0.49202	0.48803	0.48405	0.48006	0.47608	0.47210	0.46812	0.46414

z	+ 0.00	+ 0.01	+ 0.02	+ 0.03	+ 0.04	+ 0.05	+ 0.06	+ 0.07	+ 0.08	+ 0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56360	0.56749	0.57142	0.57535

⋮

z	+0.00	+0.01	+0.02	+0.03	+0.04	+0.05	+0.06	+0.07	+0.08	+0.09
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997

Scaling the Gaussian RV

$$f(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)$$

$$X \sim N(\mu = 0, \sigma^2 = 1) \sim N(0,1)$$

- $Y = \sigma X + \mu$

- $f_Y(y) =$

Connecting Gaussian to Exponential

Comparing Normal $Z \sim N(0,1)$ with Exponential $E(\lambda = \frac{1}{2})$

- $f_Z(u) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2})$ $f_E(u) = \frac{1}{2} \exp(-\frac{u}{2})$
- Let $X = Z^2$
- $f_X(u) \propto f_Z(\sqrt{u}) + f_Z(-\sqrt{u}) \propto \exp\left(-\frac{u}{2}\right) \sim E(\lambda = \frac{1}{2})$

Example – Circuit voltage noise follows $Z \sim N$

- Energy is Z^2 - cause system fail

Examples

Given $\Phi(u) \triangleq F_X(u)$ and $Q(u) = 1 - \Phi(u)$ for $X \sim N(0,1)$

- Let $Y = N(\mu = 10, \ \sigma^2 = 16)$
- Find $P\{Y > 15\}, P\{Y \leq 5\}, P\{Y^2 \geq 400\}$ and $P\{Y = 2\}$ in terms of Φ or Q

Examples

Suppose $\mu_X = 10$ and $\sigma_X^2 = 3$. Compute $P\{X < 10 - \sqrt{3}\}$ if

- X is a Gaussian RV in terms of Q
- X is a uniform RV

(Hint: $10 - \sqrt{3} \approx 8.27$)

Slido

Choose all the correct answers

(a) $\Phi(u) + Q(u) > 1$

(b) $\Phi(u) - Q(-u) = 0$

(c) $\Phi(0.5) + Q(-0.5) > 0$

(d) $\Phi(0) = 0.5$

(e) $Q(x)$ is monotonically increasing

Central Limit Theorem and Gaussian Approximation

Central Limit Theorem (CLT)

If many independent random variables are added together, and if each of them is small in magnitude compared to the sum, then the sum X has an approximately Gaussian distribution $\tilde{X}$.

- $P\{X \leq v\} \approx P\{\tilde{X} \leq v\}$
- E.g., $X \sim Binomial(n, p)$ when np and $n(1 - p)$ are not small
  - $(n, p) = (10, 0.2)$
  - $\mu_X =$
  - $\sigma_X^2 =$

  - What if np is small?
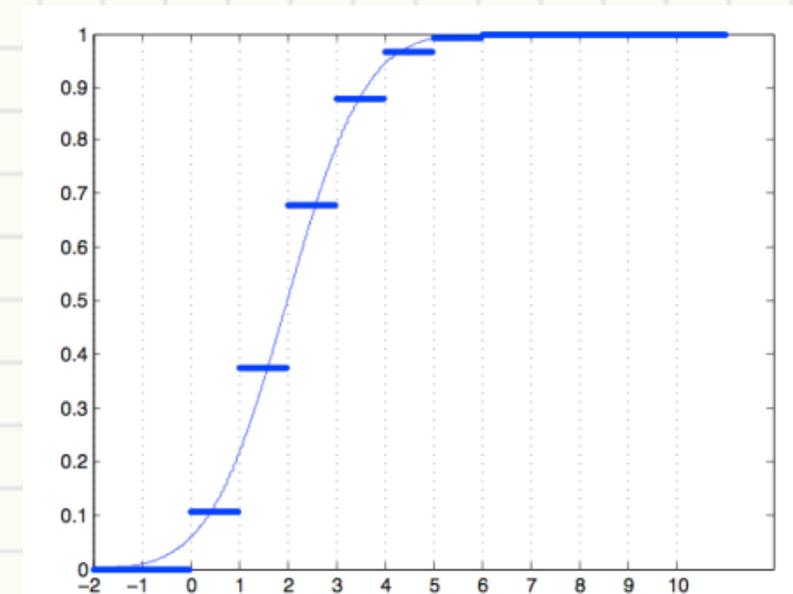
Gaussian Approximation - 1

Approximate $X \sim Bin(10, 0.2)$ with $\tilde{X} \sim N(2, 1.6)$

- $F_X(2) = F_X(2.1) = F_X(2.9)$
- $F_{\tilde{X}}(2) \neq F_{\tilde{X}}(2.9)$
- How should we approximate?

- $P\{X \leq k\} \approx$
- $P\{X < k\} \approx$
- $P\{X \geq k\} \approx$
- $P\{X > k\} \approx$