Last lecture

Uniform Distribution (Ch 3.3)

Exponential Distribution (Ch 3.4)

- Memoryless property
- Connect to Geo(p)

Poisson process (Ch 3.5)

- Motivation
- Bernoulli process to Poisson process
- Definition
- Properties

Agenda

Erlang Distribution (Ch 3.5.3) ⇒ Cont. negative binomial.

Linear Scaling (Ch 3.6.1)

- Equation and derivation
- Examples

Gaussian (normal) Distribution (Ch 3.6.2)

- Motivation and Definition
- Examples

Erlang Distribution

Definition

$T_r \sim Erlang(r, \lambda)$

Let T_r denotes the time of r^{th} count of a Poisson process

- $T_r = \sum_{i=1}^{r} U_i$, $U_i \sim Exp(\lambda)$
- $F_{T_r}^c(\mathrm{t}) = \mathrm{P}\{\mathrm{T_r} > \mathrm{t}\}$: "At most $r - 1$ count by time t"
- $F_{T_r}^c(\mathrm{t}) = \sum_{k=0}^{r-1} \underbrace{\frac{(\lambda t)^k e^{-\lambda t}}{k!}}_{P_X(k)}$ $\quad X \sim Poi(\lambda t)$
- $f_{T_r}(t) = -\frac{dF_{T_r}^c(t)}{dt} = \frac{\lambda^r t^{r-1} e^{-\lambda t}}{(r-1)!} = P_X(r-1) \cdot \lambda$
- Intuitively – Exactly $r-1$ arrival before time t, and the r^{th} event comes at t

Linear Scaling

Definition

$X = \frac{Y-b}{a}$ → x value resulting $y = u$

Let $Y = aX + b$, where X, Y are RV and a, b are constants

- $f_Y(u) = f_X\left(\boxed{\frac{u-b}{a}}\right) \times \frac{1}{\|a\|}$
- Example - $X \sim Uniform(0,1)$ $Y = 3X+2$

  $f_X(v) = \begin{cases} 1 & v \in [0,1] \\ 0 & \text{else} \end{cases}$ $f_Y(u) \begin{cases} \frac{1}{\|3\|} & u \in [2,5] \\ 0 & \text{else} \end{cases}$ ← $3([0,1])+2$

- $F_Y(u) = P\{aX + b \leq u\} = F_X\left(\frac{u-b}{a}\right)$
- $f_Y(u) = F_Y'(u) = \frac{d F_X\left(\frac{u-b}{a}\right)}{du} = f_X\left(\frac{u-b}{a}\right) \cdot \frac{1}{a} = f_X\left(\frac{u-2}{3}\right) \times \frac{1}{\|3\|}$
- $E[Y] =$ $aE[X] + b$
- $Var(Y) = a^2 Var(X)$ $\sigma_Y = a\sigma_X$

Example

Consider the temperature in Champaign

- X denotes the temperature in C (Celcius)
- Y denotes it in F (Fahrenheit)
- $Y = (1.8)X + 32$

$X = \frac{Y-32}{1.8}$

$f_Y(u) = f_X\left(\frac{u-32}{1.8}\right) \times \frac{1}{\|1.8\|}$

- Express f_Y in terms of f_X
- Find f_Y if $X \sim Uniform(15, 20)$

$f_X(v) \begin{cases} \frac{1}{5} & v \in [15, 20] \\ 0 & \text{else} \end{cases}$

$f_Y(u) = \begin{cases} \frac{1}{9} & u \in [59, 68] \\ 0 & \text{else} \end{cases}$

Example

$\mu_X = 0 \quad \sigma_X^2 = 1$

Let $X \sim Uniform(a,b)$. Find the pdf of standardized RV $\frac{X-\mu_X}{\sigma_X}$

- Recall the variance of $Uniform(0,1)$ is $\cancel{12}$ $\frac{1}{12}$

$Z = \frac{X-\mu_X}{\sigma_X}$ $\qquad f_Z(u) > 0$ if $z \in [-k, k]$

$W = Uni[0,1] \qquad Var(W) = \frac{1}{12}$

$Z = cW + d \qquad Var(Z) = c^2 Var(W) = 1$

support

$c = 2\sqrt{3}$

$k - (-k) = 2\sqrt{3} \times (1-0) \Rightarrow k = \sqrt{3}$

$$Z = cW + d \Rightarrow \underline{\underline{f_Z(u)}} = \underline{f_W\left(\frac{u-d}{c}\right)} \cdot \frac{1}{\|c\|}$$

$$f_W(v) \begin{cases} 1 & v \in [0,1] \\ 0 & \end{cases}$$

$$\boxed{\int_{-k}^{k} u^2 f_Z(u) = 1}$$

Gaussian (Normal) Distribution

Definition

A normal distribution is defined by μ_X and σ_X^2, Let $X \sim N(\mu, \sigma^2)$

- $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$

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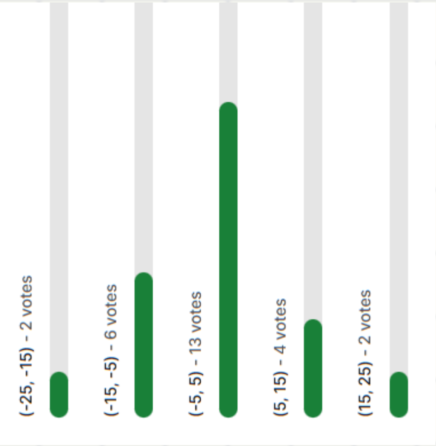
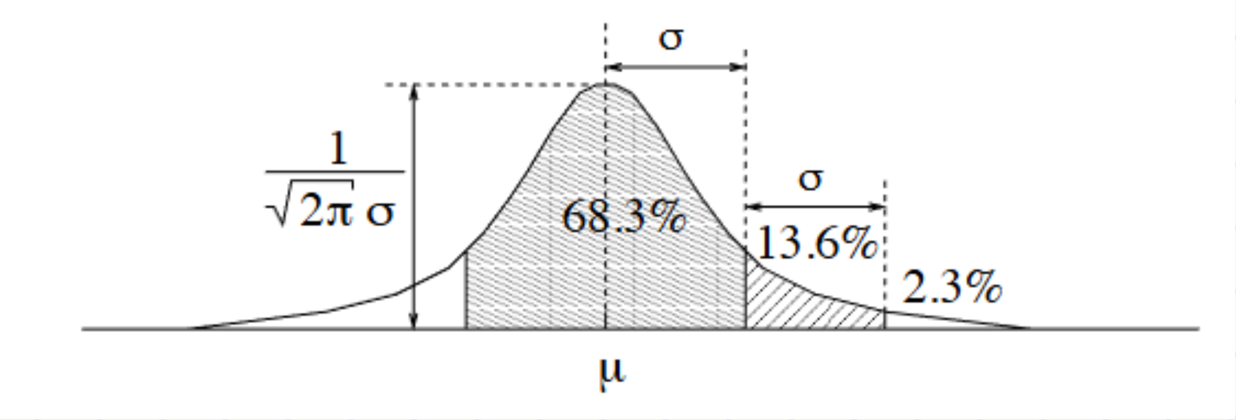
- Usage – model "Sum of many small independent events"
- E.g. Sum of many binomial distributions



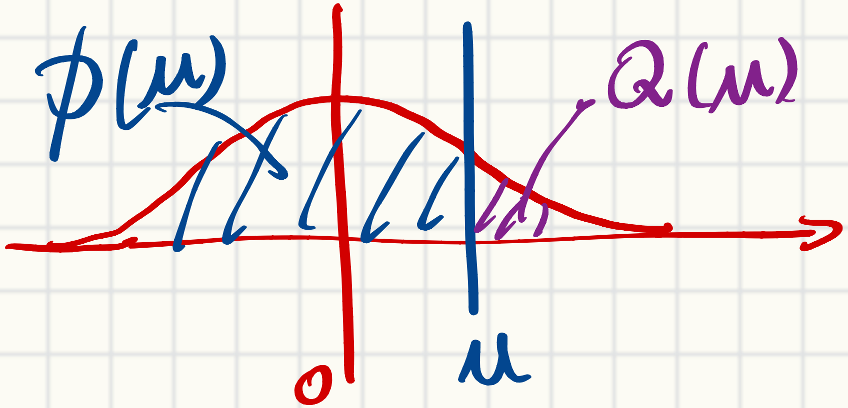


Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$X \sim N(\mu = 0, \sigma^2 = 1) \sim N(0,1) \triangleq Z$$

- $\Phi(u) \triangleq F_X(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$
- $Q(u) = 1 - \Phi(u) = F_X^c(u)$
- Pre-computed tables!



$X \sim N(0,1)$

e.g. $P\{X \le 0.5\} = \Phi(0.5)$

$P\{X \ge 2\} = Q(2)$

Φ and Q tables

z	−0.00	−0.01	−0.02	−0.03	−0.04	−0.05	−0.06	−0.07	−0.08	−0.09
-3.9	0.00005	0.00005	0.00004	0.00004	0.00004	0.00004	0.00004	0.00004	0.00003	0.00003
-3.8	0.00007	0.00007	0.00007	0.00006	0.00006	0.00006	0.00006	0.00005	0.00005	0.00005
-3.7	0.00011	0.00010	0.00010	0.00010	0.00009	0.00009	0.00008	0.00008	0.00008	0.00008
⋮										
−0.1	0.46017	0.45620	0.45224	0.44828	0.44433	0.44038	0.43644	0.43251	0.42858	0.42465
−0.0	0.50000	0.49601	0.49202	0.48803	0.48405	0.48006	0.47608	0.47210	0.46812	0.46414
z	−0.00	−0.01	−0.02	−0.03	−0.04	−0.05	−0.06	−0.07	−0.08	−0.09

$\Phi(-0.15)$

z	+ 0.00	+ 0.01	+ 0.02	+ 0.03	+ 0.04	+ 0.05	+ 0.06	+ 0.07	+ 0.08	+ 0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56360	0.56749	0.57142	0.57535
⋮										
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
z	+0.00	+0.01	+0.02	+0.03	+0.04	+0.05	+0.06	+0.07	+0.08	+0.09

Scaling the Gaussian RV

$$f(u) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2})$$

$$X \sim N(\mu = 0, \sigma^2 = 1) \sim N(0,1)$$

- $Y = \sigma X + \mu \sim N(\mu, \sigma^2)$
- $f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\left(\frac{y-\mu}{\sigma}\right)^2}{2}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Connecting Gaussian to Exponential

Exam Safe

Comparing Normal $Z \sim N(0,1)$ with Exponential $E(\lambda = \frac{1}{2})$

- $f_Z(u) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2})$ $\quad$ $f_E(u) = \frac{1}{2} \exp(-\frac{u}{2})$
- Let $X = Z^2$
- $f_X(u) \propto f_Z(\sqrt{u}) + f_Z(-\sqrt{u}) \propto \exp\left(-\frac{u}{2}\right) \sim E(\lambda = \frac{1}{2})$

Example – Circuit voltage noise follows $Z \sim N$

- Energy is Z^2 - cause system fail

Examples

Given $\Phi(u) \triangleq F_X(u)$ and $Q(u) = 1 - \Phi(u)$ for $X \sim N(0,1)$

- Let $Y = N(\mu = 10,\ \sigma^2 = 16)$ $Y = aX + b$
- Find $P\{Y > 15\}, P\{Y \leq 5\}, P\{Y^2 \geq 400\}$ and $P\{Y = 2\}$ in terms of Φ or Q

$a = 4 \quad b = 10$

$P\{Y > 15\} = P\{4X + 10 > 15\}$

$= P\{X > \frac{5}{4}\} = Q(\frac{5}{4})$

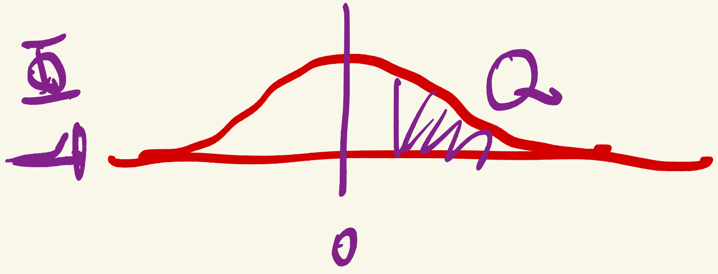
$P\{Y \leq 5\} = P\{4X + 10 \leq 5\} = P\{X < -\frac{5}{4}\}$

$= \Phi(-\frac{5}{4})$

$$P\{Y^2 \ge 400\} = P\{Y \le -20\} + P\{Y \ge 20\}$$

$$= P\{4X+10 \le -20\} + P\{4X+10 \ge 20\}$$

$$= \Phi(-7.5) + Q(2.5) \approx Q(2.5)$$



compare if ans. B

$$\Phi(3) + \underbrace{Q(2.5)}_{= \Phi(-2.5)}$$

$$= \Phi(3) + \Phi(-2.5)$$