Last lecture

Continuous RV (Ch 3)

- Motivation
- Cumulative Distribution Function (Ch 3.1)
- Examples
- CDF to PMF and probabilistic density function (PDF)

Continuous RV & Probability Density Function (Ch 3.2)

- Definition
- Facts

Uniform Distribution (Ch 3.3)

Agenda

Uniform Distribution (Ch 3.3)

Exponential Distribution (Ch 3.4) → Cont. Geo.

- Memoryless property
- Connect to Geo(p)

Poisson process (Ch 3.5)

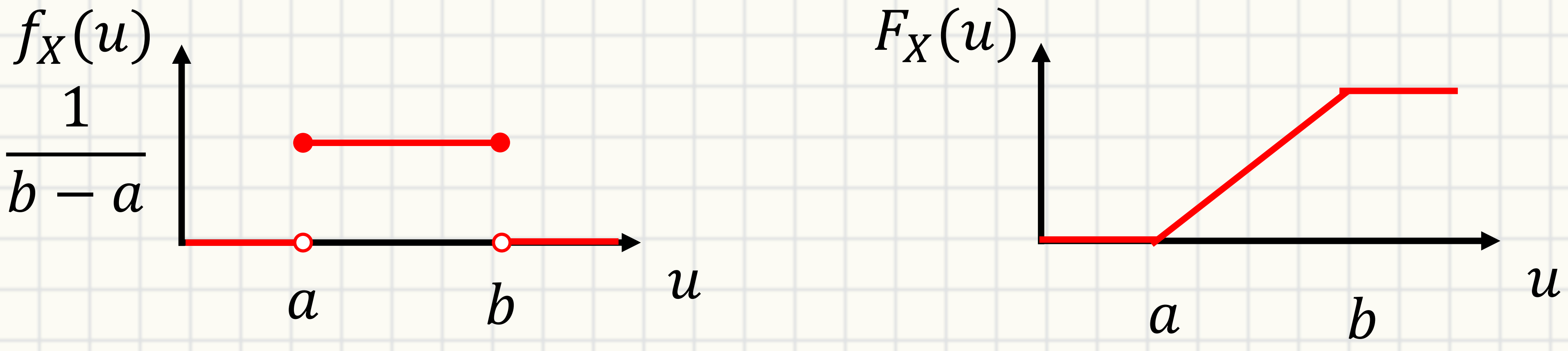
- Motivation
- Bernoulli process to Poisson process
- Definition
- Properties

Erlang Distribution (Ch 3.5.3) ⇒ Cont. N.B.

Uniform Distribution

Uniform Distribution

$$f_X(u) = \begin{cases} \frac{1}{b-a} & \text{if } a \le u \le b \\ 0 & \text{else} \end{cases}$$



Properties

$$f_X(u) = \begin{cases} \frac{1}{b-a} & \text{if } a \le u \le b \\ 0 & \text{else} \end{cases}$$

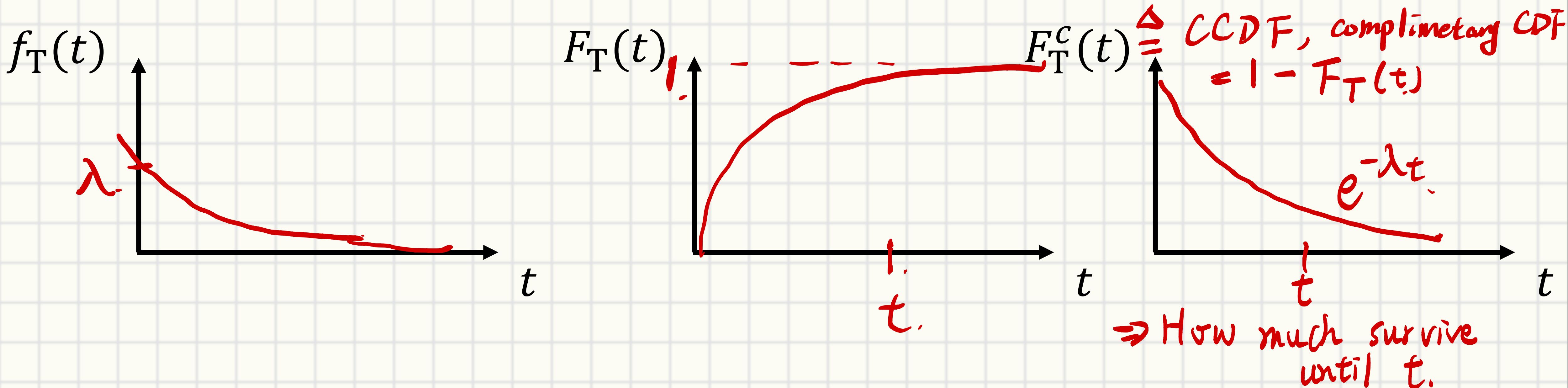
- $E[X] = \int_{-\infty}^{\infty} u f_X(u)\, du = \int_a^b \frac{u}{b-a} du = \frac{b-a}{2}$
- $E[X^2] = \int_{-\infty}^{\infty} u^2 f_X(u)\, du = \int_a^b \frac{u^2}{b-a} du = \frac{b^2+ab+a}{3}$
- $Var(X) = \frac{(a-b)^2}{12}$
- Special case, when $(a,b) = (0,1)$
  - k^{th} moment $E[X^k] = \frac{1}{\mathrm{k+1}}$
  - $Var(X) = \frac{1}{12}$

Exponential Distribution

Exponential Distribution

Motivation – System life for failure rate λ $\int_0^t f_T(t)$

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases} \qquad F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$



Properties

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

Int. by part

- $E[T^n] \quad = \int_0^\infty t^n \lambda e^{-\lambda t} dt$

$$= -t^n e^{-\lambda t} \Big|_0^\infty + \int_0^\infty n t^{n-1} e^{-\lambda t} dt$$

$$= 0 + \frac{n}{\lambda} \int_0^\infty t^{n-1} \lambda e^{-\lambda t} dt = \frac{n}{\lambda} E[T^{n-1}]$$

I>p in Geo.

- $E[T] = \frac{1}{\lambda} \qquad E[T^2] = \frac{2}{\lambda^2} \qquad E[T^n] = \frac{n!}{\lambda^n}$

- $Var(T) = E[T^2] - \mu_T^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Examples

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

Let $T \sim Exp(\lambda = ln2)$, find $P\{T \geq t\}$ and $P(T \leq 1|T \leq 2)$

$$P(T \geq t) = F_T^c(t) = e^{-\lambda t} = e^{-\ln 2 t}$$

$$= 2^{-t}$$

$$\frac{P\{T \leq 1, \cancel{T \leq 2}\}}{P\{T \leq 2\}} = \frac{F_T(1)}{F_T(2)} = \frac{1 - e^{-\lambda}}{1 - e^{-2\lambda}} = \frac{2}{3}$$

$\lambda = \ln 2$

Memoryless Property

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

$$P\{T \geq t\} = e^{-\lambda t}$$

- $P\{T \geq s + t | T \geq s\} = \dfrac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}$

  $= P\{T \geq t\}$

- If T is the system lifetime

$T \geq s+t \,|\, T \geq s$: survive another t

$= T \geq t$: survive first t.

Connecting $\boldsymbol{Exp}$ with $\boldsymbol{Geo}$

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

Summary - $F_L(\lfloor \frac{c}{h} \rfloor) \to F_T(c)$ when $h \to 0$

- $L \sim Geo(p = \lambda h)$ Time interval
- $T \sim Exp(\lambda = \lambda)$

A lightbulb of average lifetime 1000hrs

- Failed hour = $L \sim Geo(p = \frac{1}{1000})$
- Let's assume it will only fail at start of each ticks h hours (e.g., sec, $h = 1/3600$)
- Failed ticks = $L_h \sim Geo(p_h = \frac{1}{1000} \times h)$

Connecting Exp with Geo

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

A lightbulb of average lifetime 1000hrs

- Failed hour = $L \sim Geo(p = \frac{1}{1000})$
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- Failed ticks = $L_h \sim Geo(p = \frac{1}{1000} \times h)$

survival time

- $P\{L_h h > c\} = P\left\{L_h > \frac{c}{h}\right\} = F^c_{L_h}\left(\left\lfloor \frac{c}{h} \right\rfloor\right) = (1-p)^{\left\lfloor \frac{c}{h} \right\rfloor}$

$$= (1 - \lambda h)^{\left\lfloor \frac{c}{h} \right\rfloor} = \lim_{h \to 0} e^{-\lambda c} = F^c_T(c)$$

Slido – Waiting for the bus

#2045963

Say a bus comes to the stop every 10 minutes in average

- Expected waiting time follows $T \sim Exp(\lambda = \frac{1}{10})$
- Alice knows the time last bus leave at $t = 0$, what's her best strategy to arrive t^* at the stop minimizing her waiting time T'?

$$= P\{T \geq T' + t^* \mid T \geq t^*\}$$

$$= P\{T \geq T'\}$$

(A) $t^* = 0$

(B) $t^* = 5$

(C) $t^* = 10$

(D) Doesn't matter

Poisson Process

Motivation

- Model the process of customer coming inside a coffee shop
- The process of incoming calls for customer service
  - If a customer (call) only comes at the start. of every minutes -> Bernoulli process
  - But what if we want to model the time
  - Define a small h between Bernoulli trials

Bernoulli Process $\xrightarrow{h \to 0,\ \lambda = \frac{p}{h}}$ Poisson Process

- Assume each trial takes h time to complete

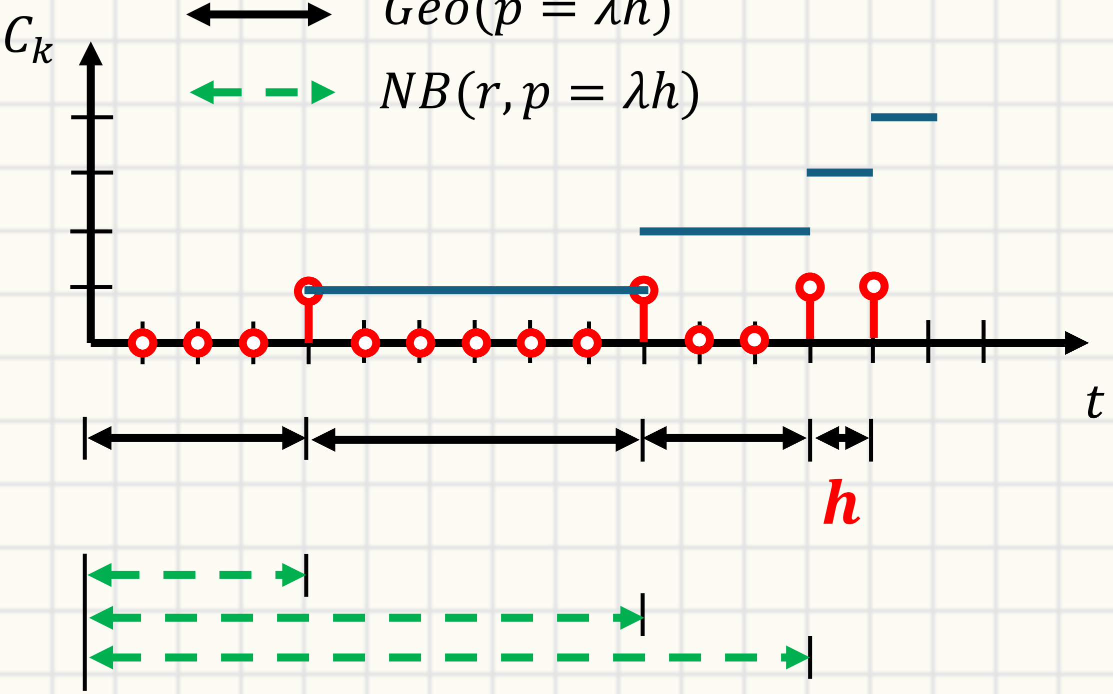
$h \to 0$

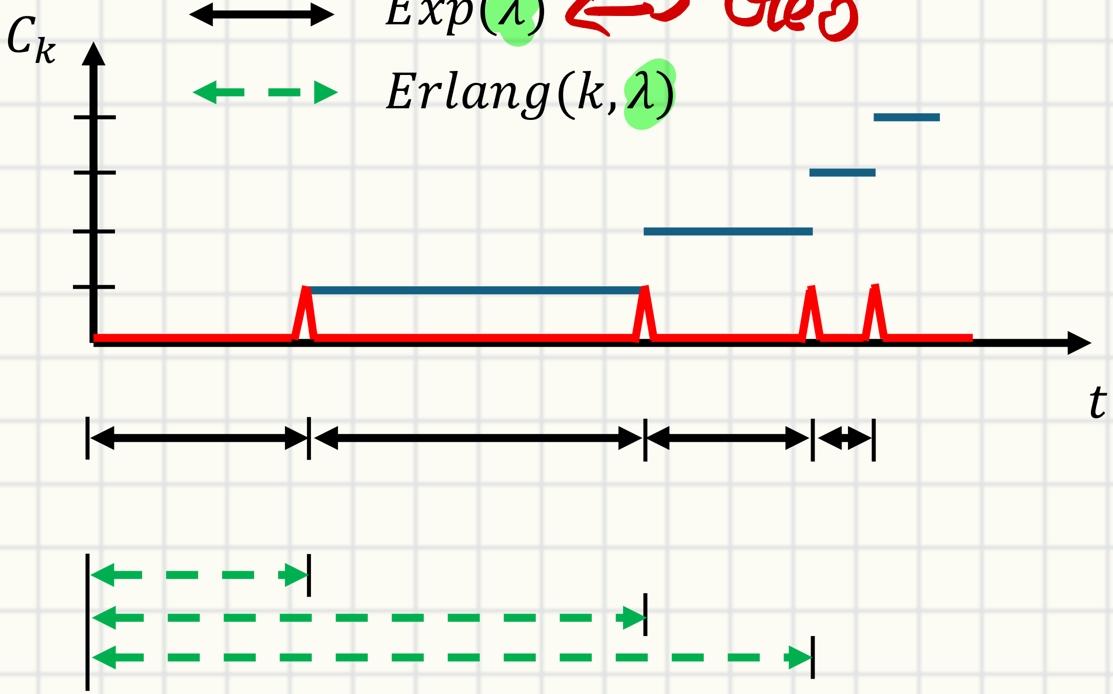
n in Bin. $\to \infty$

$n = \frac{t}{h}$

	$\xrightarrow{h \to 0,\ \lambda = \frac{p}{h}}$	
$Bern(p)$		
$Bin\left(n = \left\lfloor \frac{t}{h} \right\rfloor, p = \lambda h\right)$		$Poi(\lambda = \lambda t)$
$Geo(p = \lambda h)$		$Exp(\lambda)$ $\longleftrightarrow$ Geo
$NB(r, p = \lambda h)$		$Erlang(k, \lambda)$

Avg. err. ⟵ λ ; λ ⟶ failing rate





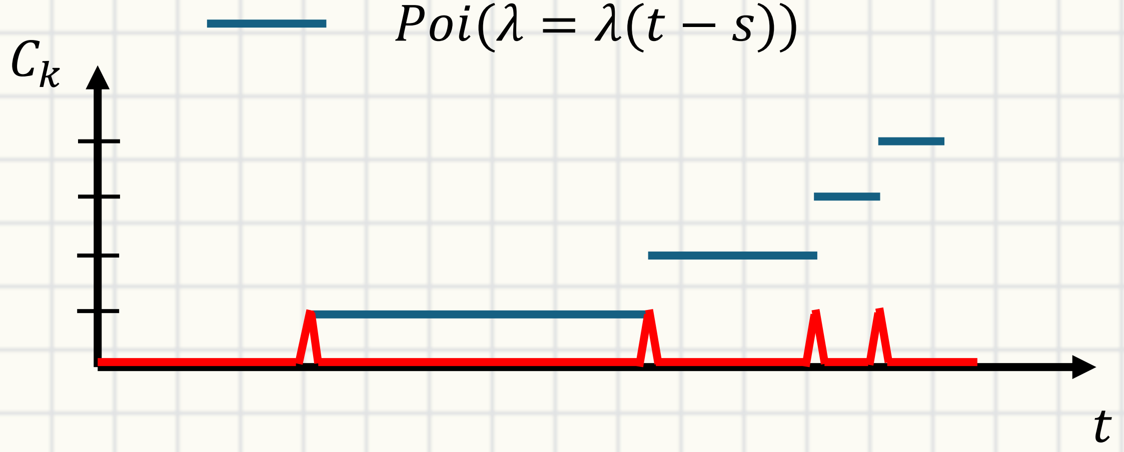
Definition

A Poisson process with rate λ is a random counting process $N = (N_t : t \geq 0)$ s.t.

Avg # error rate

- $N_t - N_s$ follows Poisson distribution $Poi(\lambda = \lambda(t-s))$
- For $0 \leq t_1 \leq t_2 \ldots \leq t_k$, $N_{t_k} - N_{t_{k-1}}$ the increments are independent with each other

none overlap. seg.



Example

Consider a Poisson process with rate $\lambda > 0$ in time interval $[0, T]$

- X is the total number of count during $[0, T]$
- X_1 is the count during $[0, \tau]$, $0 < \tau < T$
- X_2 is the count during $[\tau, T]$ none overlap

  $\Rightarrow X_1$ & X_2 are indepent.

- Solve $P\{X = n\}$, $P\{X_1 = i\}$ and $P\{X_2 = j\}$

$X \sim Poi(\lambda = \lambda T)$

$$P_X(n) = \frac{(\lambda T)^n e^{-(\lambda T)}}{n!}$$

$$\frac{(\lambda \tau)^i e^{-(\lambda \tau)}}{i!}$$

$$\frac{[\lambda(T-\tau)]^j e^{-[\lambda(T-\tau)]}}{j!}$$

Example

Consider a Poisson process with rate $\lambda > 0$ in time interval $[0, T]$

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- X_1 is the count during $[0, \tau]$, $0 < \tau < T$
- X_2 is the count during $[\tau, T]$

- Let $n = i + j$
- Solve $P(X = n | X_1 = i)$

$$= P\{X_1 = i, X_2 = j \mid X_1 = i\}$$

$$= \frac{P\{X_1 = i\} P\{X_2 = j\}}{P\{X_1 = i\}} = P_{X_2}(j)$$

Example

Consider a Poisson process with rate $\lambda > 0$ in time interval $[0, T]$

- X is the total number of count during $[0, T]$
- X_1 is the count during $[0, \tau]$, $0 < \tau < T$
- X_2 is the count during $[\tau, T]$

- Let $n = i + j$
- Solve $P(X_1 = i | X = n)$

Example

Calls arrive to a support center at rate $\lambda = 2$ calls per minute.
Let N_t denotes the number of calls until time t (mins)

- $N_t \sim$ $Poi(\lambda = 2t)$
- $P_{N_t}(k) =$ $\longrightarrow$ $\dfrac{(2t)^k e^{-(2t)}}{k!}$
- $P\{$ No calls arrive in the first 3.5 minutes $\}$
- $P\{$ The third call arrives after time t = 3.$\}$ $\leftarrow$ < 3 calls before $t=3$

$N_{3.5} \sim Poi(\lambda = 2 \times 3.5) = Poi(\lambda = 7)$

$P_{N_{3.5}}(0) = \dfrac{7^0 e^{-7}}{0!} = e^{-7}$

$\sum_{k=0}^{2} P_{N_3}(k)$