Last lecture

Binary Hypothesis Testing (Ch 2.11)

- Likelihood Ratio Test (LRT)
- Maximum likelihood decision rule
- Maximum A Posteriori (MAP) decision rule
- Examples

Union Bound/ Reliability (Ch 2.12)

Agenda

Continuous RV (Ch 3)

- Motivation
- Cumulative Distribution Function (Ch 3.1)
- Examples
- CDF to PMF and probabilistic density function (PDF)

Continuous RV & Probability Density Function (Ch 3.2)

- Definition
- Facts

Uniform Distribution (Ch 3.3)

Continuous RV

Motivation

Tired of coin toss/ win-lose?

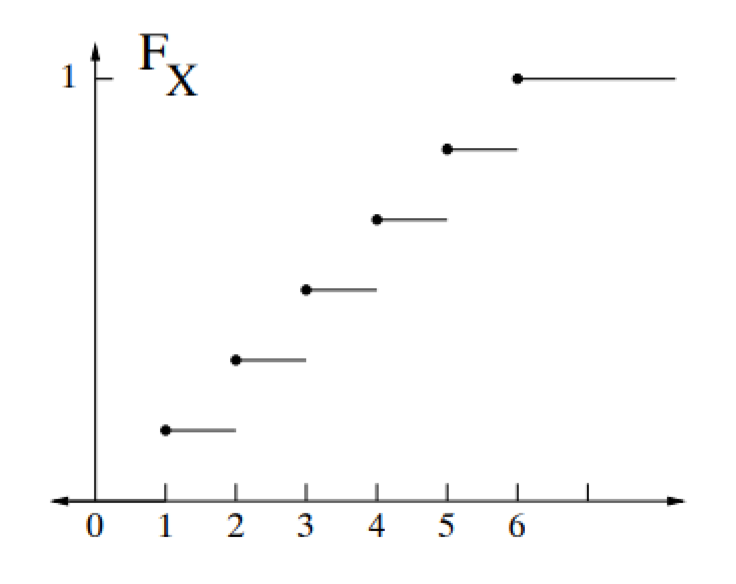
- Real-world is continuous
- Time, space, height, weight, colors, etc.
- But how do we define a continuous RV?
- Recall for prob. space $(\Omega, \mathcal{F}, P)$
  - X maps $\omega \in \Omega$ to $\mathbb{R}$ (coated die)
  - What if ω is continuous?
  - Discrete $\{\omega: X(\omega) = c\}$ -> Continuous $\{\omega: X(\omega) \leq c\}$
  - $F_X(c) = P\{\omega: X(\omega) \leq c\} = P\{X \leq c\}$

Cumulative Distribution Function (CDF

Recall for prob. space $(\Omega, \mathcal{F}, P)$

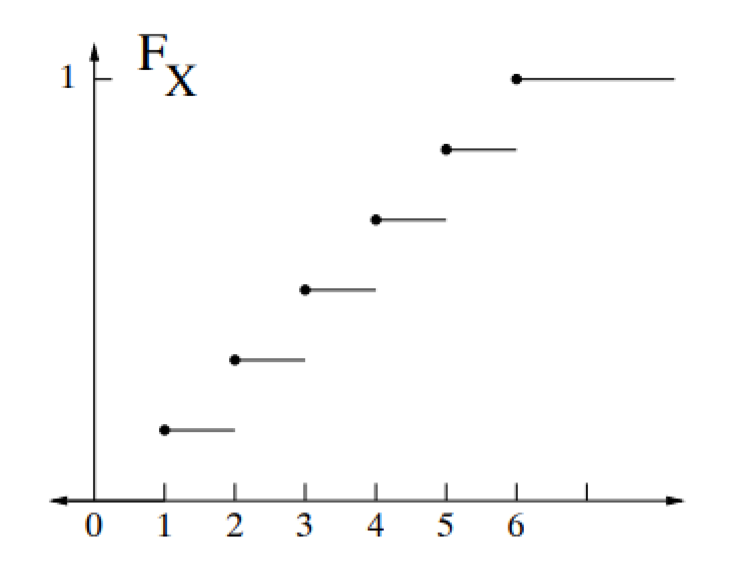
- X maps $\omega \in \Omega$ to $\mathbb{R}$ (coated die)
- PMF $\{\omega: X(\omega) = c\}$ -> CDF $\{\omega: X(\omega) \leq c\}$
- $F_X(c) = P\{\omega: X(\omega) \leq c\} = P\{X \leq c\}$

CDF of a fair die roll



Recall – Left limit and Right limit

- $F_X(x-) = \lim_{\substack{y \to x \\ y<x}} F_X(y)$ $\qquad F_X(x+) = \lim_{\substack{y \to x \\ y>x}} F_X(y)$
- $F_X(x) \triangleq F_X(x+)$
- $F_X(2+) =$
- $\Delta F_X(x) = F_X(x) - F_X(x-)$
- $P\{X \in (a,b]\} =$



Examples

- Find all u where $P\{X = u\} > 0$
- Find $P\{X \leq 0\}$
- Find $P\{X < 0\}$

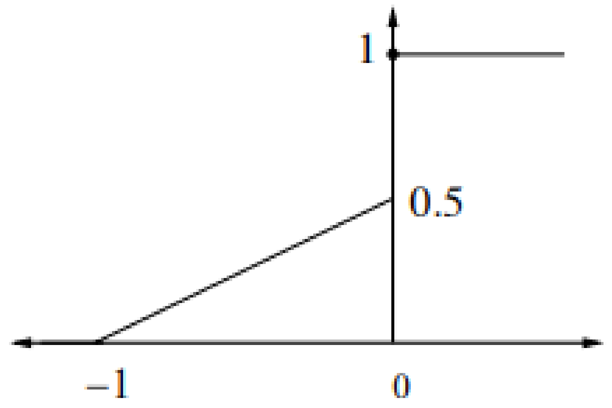


Figure 3.2: An example of a CDF.

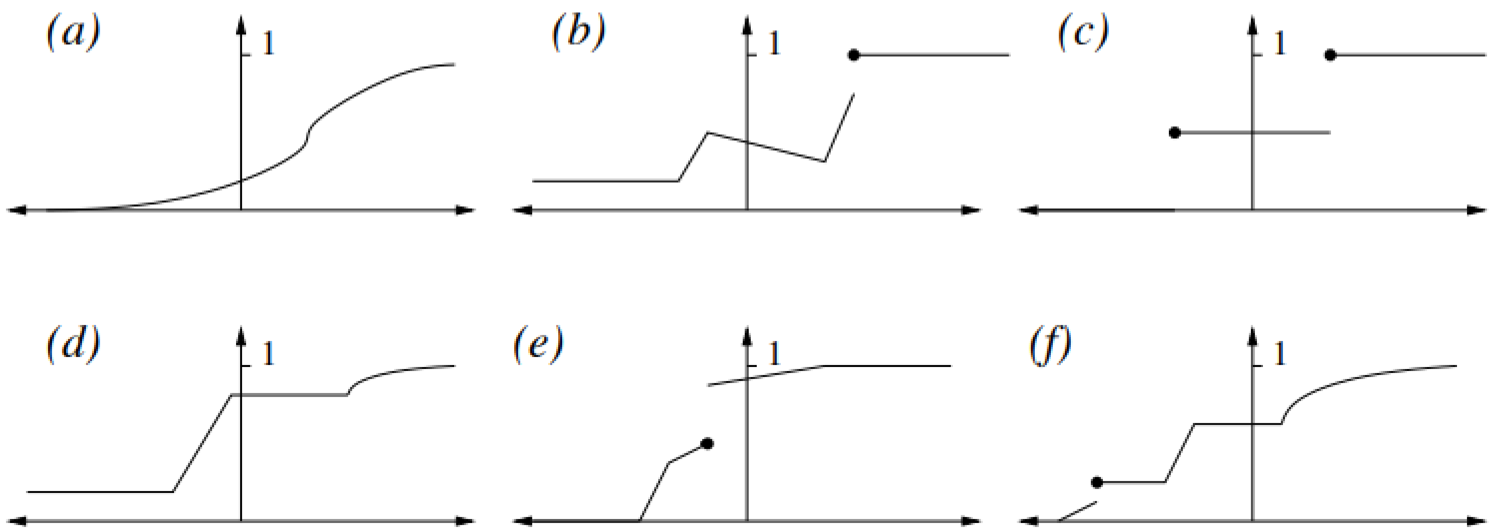
CDF Properties

A function F is a CDF of some RV iif

- F is none-decreasing

- $\lim_{c\to\infty} F(c) = 1$ and $\lim_{c\to-\infty} F(c) = 0$

- F is right-continuous



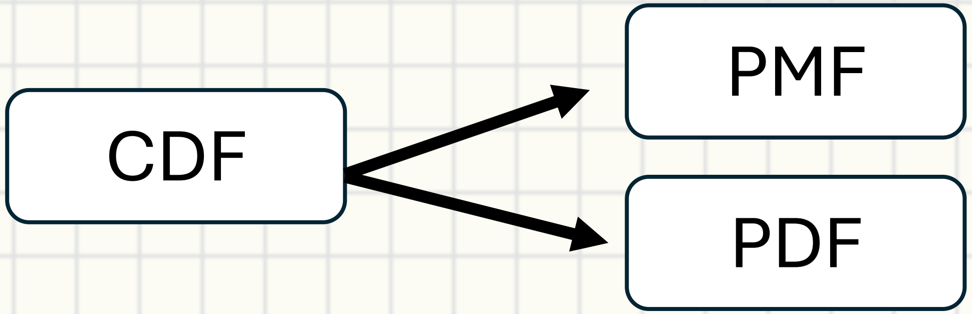
CDF to PMF and PDF

For discrete RV

- $F_X(c) =$
- $p_X(u) =$

For continuous RV

- $F_X(c) =$
- $f_X(u) =$
- $f_X(u)$ is called "   " (PDF)



Continuous RV (Ch 3.2)

Continuous RV and PDF

X is a continuous RV if its pdf f_X follows

- $F_X(c) = \int_{-\infty}^{c} f_X(u)du$ for all $c \in \mathbb{R}$
- Support – $\{u: f_X(u) > 0\}$
- If $F_X(c)$ is continuous and differentiable, $f_X = F_X{}'$

  - Since there is no jump in $F_X(c)$, $P\{X = c\} =$

- $P\{a < X \leq b\} = F_X(b) - F_X(a) =$

- $\int_{-\infty}^{\infty} p_X(u)du$

Why P "density" F

By definition, $f_X = F'_X$

$$f_X(u_0) = \lim_{h\to 0} \frac{F(u_0+h) - F(u_0)}{h}$$

$$f_X(u_0) = \lim_{h\to 0} \frac{F(u_0) - F(u_0-h)}{h}$$

$$f_X(u_0) = \lim_{h\to 0} \frac{F(u_0+h) - F(u_0-h)}{2h}$$

- Let $\epsilon = 2h > 0$
- $f_X(u_0) = \lim_{h\to 0} \frac{F\left(u_0+\frac{\epsilon}{2}\right) - F\left(u_0-\frac{\epsilon}{2}\right)}{\epsilon}$
- $P\left\{u_0 - \frac{\epsilon}{2} < X < u_0 + \frac{\epsilon}{2}\right\} = \epsilon f_X(u_0) + O(\epsilon)$
- "Density of the probability"

Expectation and Variance

- $\mu_X = E[X] = \int_{-\infty}^{\infty} u f_X(u) du$
- LOTUS still applies, $E[g(x)] = \int_{-\infty}^{\infty} g(u) f_X(u) du$
- E.g. $E[aX^2 + bX + c] = aE[X^2] + bE[X] + c$

- $\sigma_X^2 = Var(X) = E[(X - \mu_X)^2] = E[X^2] - \mu_X^2$

Example

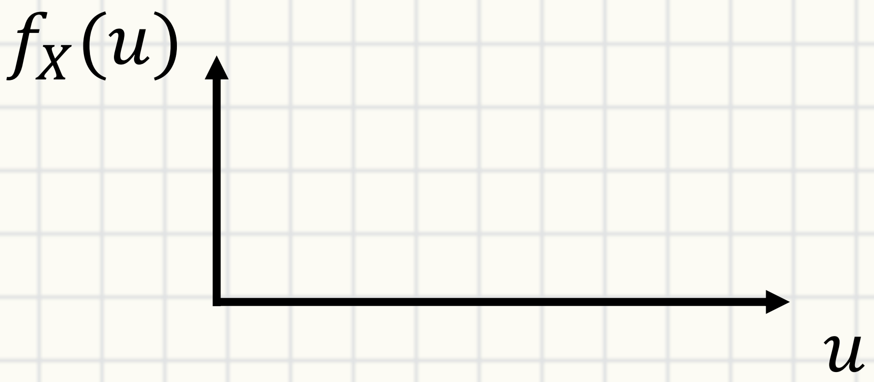
- $f_X(u) = \begin{cases} A(1-u^2) & if \quad -1 \le u \le 1 \\ 0 & else \end{cases}$

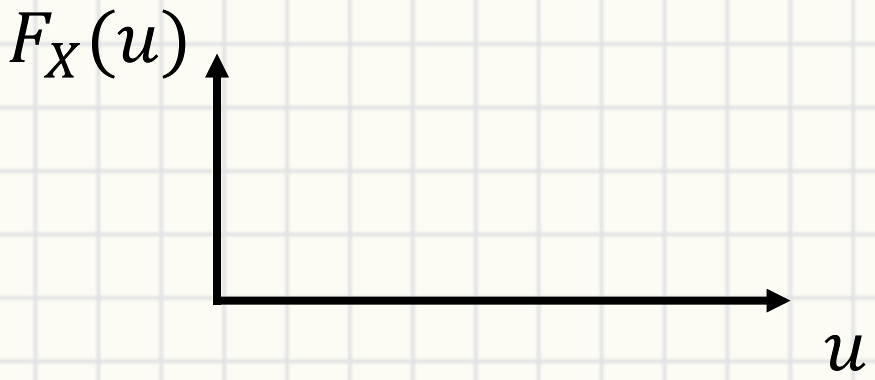
- Find A, $P\{-0.5 < X < 1.5\}$, F_X, μ_X, σ_X^2

Uniform Distribution

Uniform Distribution

$$f_X(u) = \begin{cases} \frac{1}{b-a} & \text{if } a \le u \le b \\ 0 & \text{else} \end{cases}$$





Properties

$$f_X(u) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq u \leq b \\ 0 & \text{else} \end{cases}$$

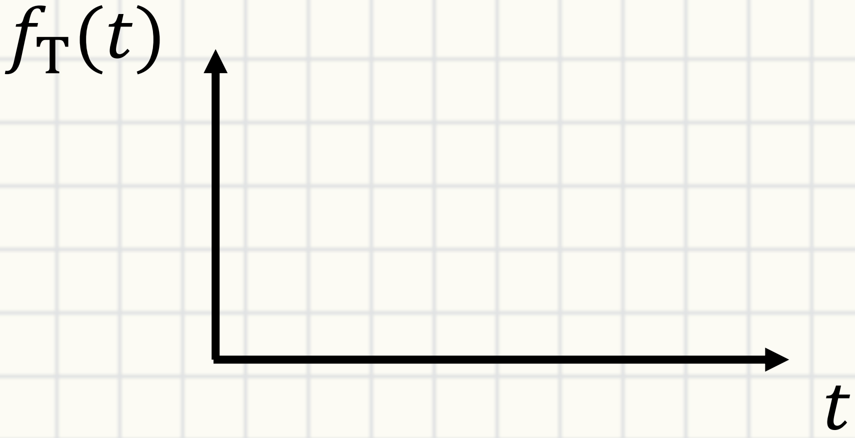
- $E[X] = \int_{-\infty}^{\infty} u f_X(u)\, du =$
- $E[X^2] = \int_{-\infty}^{\infty} u^2 f_X(u)\, du =$
- $Var(X) =$
- Special case, when $(a, b) = (0,1)$
  - k^{th} moment $E[X^k] =$
  - $Var(X) =$

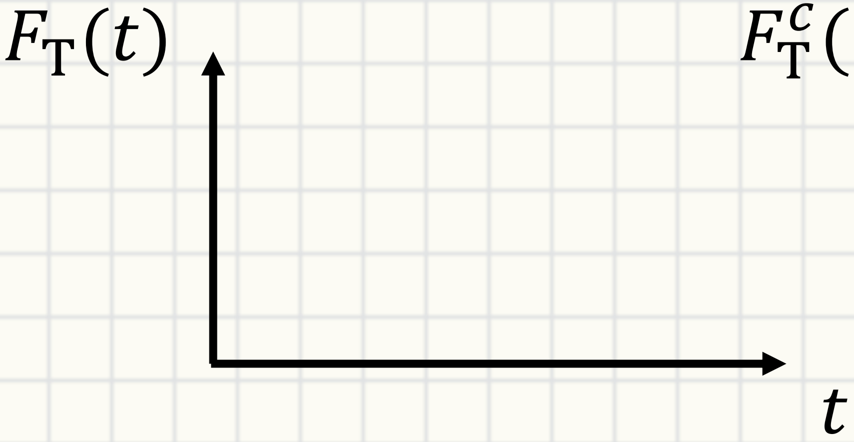
Exponential Distribution

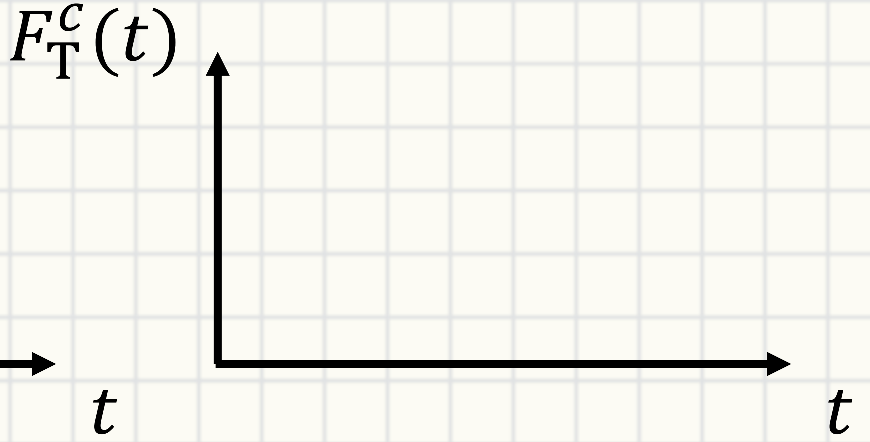
Exponential Distribution

Motivation – System life for failure rate λ

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases} \qquad F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$







Properties

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

- $$E[T^n] = \int_0^\infty t^n \lambda e^{-\lambda t} dt$$
  $$= -t^n e^{-\lambda t} \Big|_0^\infty + \int_0^\infty n t^{n-1} e^{-\lambda t} dt$$
  $$= 0 + \frac{n}{\lambda} \int_0^\infty t^{n-1} \lambda e^{-\lambda t} dt = \frac{n}{\lambda} E[T^{n-1}]$$

- $E[T] = \frac{1}{\lambda}$ $\qquad E[T^2] = \frac{2}{\lambda^2}$ $\qquad E[T^n] = \frac{n!}{\lambda^n}$

- $Var(T) =$

Examples

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

Let $T \sim Exp(\lambda = ln2)$, find $P\{T \geq t\}$ and $P(T \leq 1|T \leq 2)$

Memoryless Property

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

$$P\{T \geq t\} = e^{-\lambda t}$$

- $P\{T \geq s + t | T \geq s\} =$

- If T is the system lifetime