Last lecture

Binary Hypothesis Testing (Ch 2.11)
- Likelihood Ratio Test (LRT)
- Maximum likelihood decision rule
- Maximum A Posteriori (MAP) decision rule
- Examples

Union Bound/ Reliability (Ch 2.12)

Agenda

Continuous RV (Ch 3)

- Motivation
- Cumulative Distribution Function (Ch 3.1)
- Examples
- CDF to PMF and probabilistic density function (PDF)

Continuous RV & Probability Density Function (Ch 3.2)

- Definition
- Facts

Uniform Distribution (Ch 3.3)

Continuous RV

Motivation

Tired of coin toss/ win-lose?

- Real-world is continuous
- Time, space, height, weight, colors, etc.
- But how do we define a continuous RV?
- Recall for prob. space $(\Omega, \mathcal{F}, P)$
  - X maps $\omega \in \Omega$ to $\mathbb{R}$ (coated die)
  - What if ω is continuous?
  - Discrete $\{\omega : X(\omega) = c\}$ -> Continuous $\{\omega : X(\omega) \leq c\}$
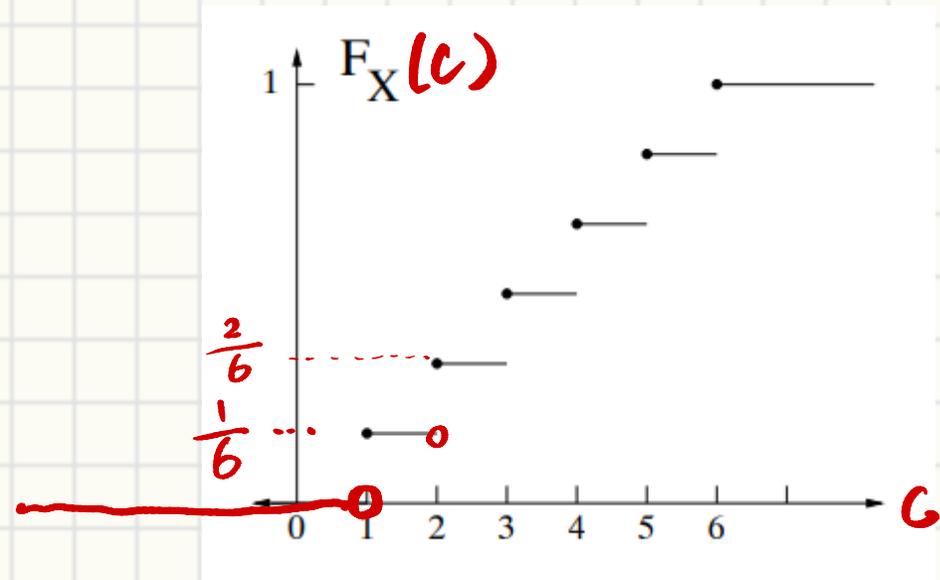 - $F_X(c) = P\{\omega : X(\omega) \leq c\} = P\{X \leq c\}$

Cumulative Distribution Function.

Cumulative Distribution Function (CDF

Recall for prob. space $(\Omega, \mathcal{F}, P)$

- X maps $\omega \in \Omega$ to $\mathbb{R}$ (coated die)
- PMF $\{\omega : X(\omega) = c\}$ -> CDF $\{\omega : X(\omega) \leq c\}$
- $F_X(c) = P\{\omega : X(\omega) \leq c\} = P\{X \leq c\}$

CDF of a fair die roll

Recall – Left limit and Right limit

- $F_X(x-) = \lim\limits_{\substack{y \to x \\ y < x}} F_X(y)$ $\qquad\qquad$ $F_X(x+) = \lim\limits_{\substack{y \to x \\ y > x}} F_X(y)$
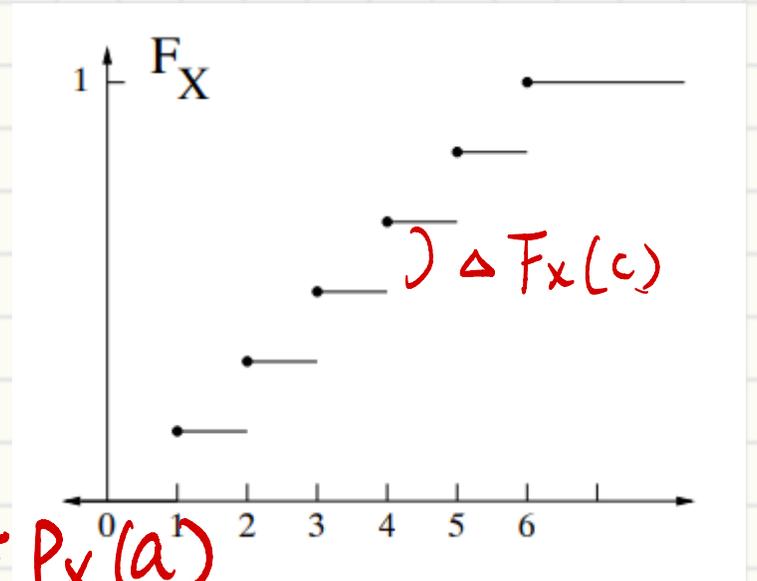
- $F_X(x) \triangleq F_X(x+)$
- $F_X(2+) =$
- $\Delta F_X(x) = F_X(x) - F_X(x-) = P_X(x)$

- $P\{X \in (a, b]\} =$

$$F_X(b) - F_X(a)$$

$$) \Delta F_X(c)$$

$$P\{X \in [a, b]\} = F_X(b) - F_X(a) + P_X(a)$$

$$= F_X(b) - F_X(a-)$$

Examples

- Find all u where $P\{X = u\} > 0$
- Find $P\{X \leq 0\}$
- Find $P\{X < 0\}$

$\mu = 0$ $\quad$ $P\{X = 0\} = 1 - 0.5 = 0.5$

$$P\{X = \mu\} = F_X(\mu) - F_X(\mu^-)$$

$$> 0$$

$\Rightarrow$ at μ, F_X is non-continuous

$$P\{X \leq 0\} = F_X(0) = 1$$

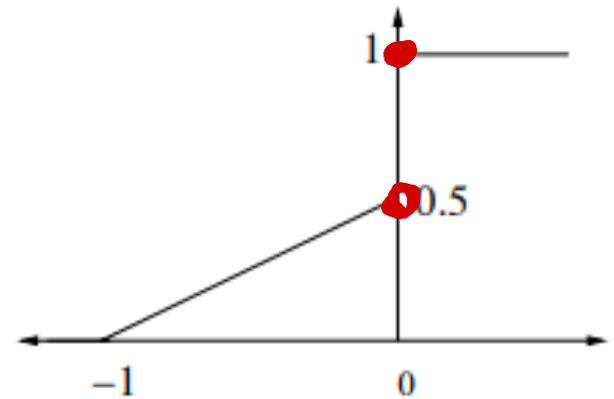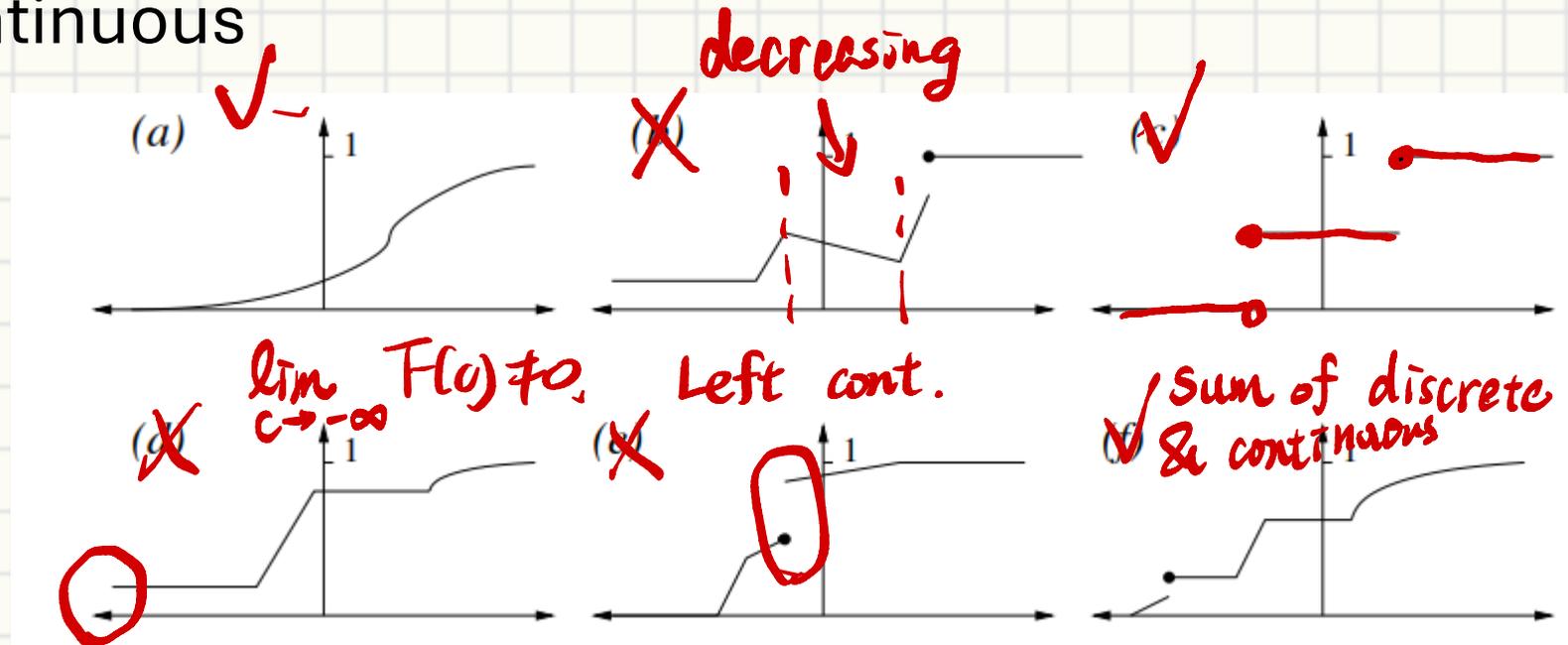$$P\{X < 0\} = F_X(0) - P_X(0)$$

$$= F_X(0^-) = 0.5$$

Figure 3.2: An example of a CDF.

CDF Properties

A function F is a CDF of some RV iif

- F is none-decreasing   ✓ $F_X(c) \in [0, 1]$

- $\lim\limits_{c \to \infty} F(c) = 1$ and $\lim\limits_{c \to -\infty} F(c) = 0$

- F is right-continuous

(a) ✓

(X) decreasing

(✓)

(X) $\lim\limits_{c \to -\infty} F(c) \neq 0$

(X) Left cont.
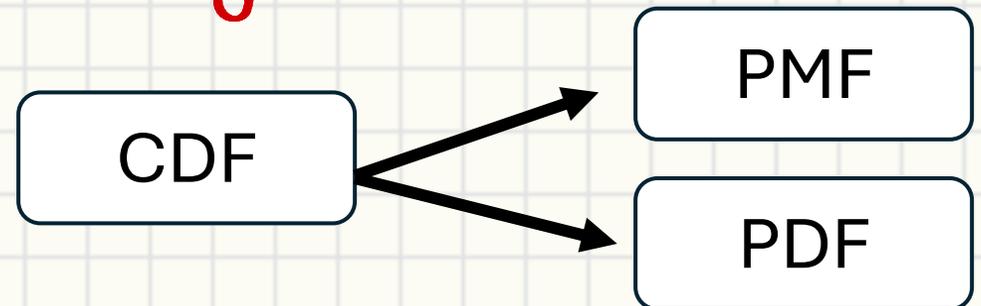
✓ Sum of discrete & continuous

CDF to PMF and PDF

For discrete RV

- $F_X(c) = \sum_{u \leq c} p_X(u)$
- $p_X(u) = F_X(u) - F_X(u-)$

For continuous RV

- $F_X(c) = \int_{-\infty}^{c} f_X(u)\, du$

- $f_X(u) = F_X'(u)$

- $f_X(u)$ is called " probability density function " (PDF)

CDF → PMF

CDF → PDF

Continuous RV (Ch 3.2)

Continuous RV and PDF

X is a continuous RV if its pdf f_X follows

- $F_X(c) = \int_{-\infty}^{c} f_X(u)\,du$ for all $c \in \mathbb{R}$
- Support $- \{u : f_X(u) > 0\}$
- If $F_X(c)$ is continuous and differentiable, $f_X = F_X{'}$

  - Since there is no jump in $F_X(c)$, $P\{X = c\} =$

- $P\{a < X \leq b\} = F_X(b) - F_X(a) = \int_{a}^{b} f_X(u)\,du$

- $\int_{-\infty}^{\infty} p_X(u)\,du = F_X(\infty) - F_X(-\infty) = 1 - 0 = 1$

Why P "density" F

By definition, $f_X = F_X'$

Not Tested

$$f_X(u_0) = \lim_{h \to 0} \frac{F(u_0 + h) - F(u_0)}{h}$$

$$f_X(u_0) = \lim_{h \to 0} \frac{F(u_0) - F(u_0 - h)}{h}$$

$$f_X(u_0) = \lim_{h \to 0} \frac{F(u_0 + h) - F(u_0 - h)}{2h}$$

- Let $\epsilon = 2h > 0$

- $f_X(u_0) = \lim_{\epsilon \to 0} \dfrac{F\left(u_0 + \frac{\epsilon}{2}\right) - F\left(u_0 - \frac{\epsilon}{2}\right)}{\epsilon}$

- $P\left\{u_0 - \dfrac{\epsilon}{2} < X < u_0 + \dfrac{\epsilon}{2}\right\} = \epsilon f_X(u_0) + O(\epsilon)$

- "Density of the probability"

Expectation and Variance

- $\mu_X = E[X] = \int_{-\infty}^{\infty} u f_X(u) du$
- LOTUS still applies, $E[g(x)] = \int_{-\infty}^{\infty} g(u) f_X(u) du$
- E.g. $E[aX^2 + bX + c] = aE[X^2] + bE[X] + c$

- $\sigma_X^2 = Var(X) = E[(X - \mu_X)^2] = E[X^2] - \mu_X^2$

$$\int_{-\infty}^{\infty} u^2 f_X(u) du$$

Example

- $f_X(u) = \begin{cases} A(1 - u^2) & if \quad -1 \leq u \leq 1 \\ 0 & else \end{cases}$

- Find A, $\quad P\{-0.5 < X < 1.5\}$, $\quad F_X$, $\quad \mu_X$, $\quad \sigma_X^2$

$$\int_{-\infty}^{\infty} f_X(u)\, du = 1 = \int_{-1}^{1} A(1 - u^2)\, du$$

$$= Au - \frac{A}{3} u^3 \Big|_{u=-1}^{u=1}$$

$$= A \frac{4}{3} = 1 \quad \Rightarrow \quad A = \frac{3}{4}$$

$$P\{-0.5 \leq X \leq 1.5\} = P\{-0.5 \leq X \leq 1\}$$

$$= P\{-0.5 < X < 1\}$$

$$= \int_{-0.5}^{1} \frac{3}{4}(1-\mu^2)\,d\mu$$

$$\mu = -1$$
$$-1 + \frac{1}{3} = -\frac{2}{3}$$

$$Fx = \int_{-\infty}^{\infty} \frac{3}{4}(1-\mu^2)\,d\mu = \frac{3}{4}\left(\mu - \frac{\mu^3}{3} + C\right)$$

$$Fx(-1) = 0. \qquad C = \frac{2}{3}$$
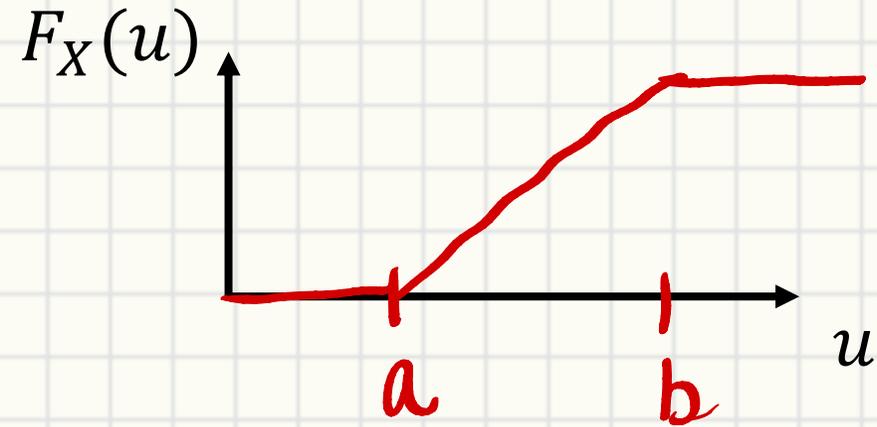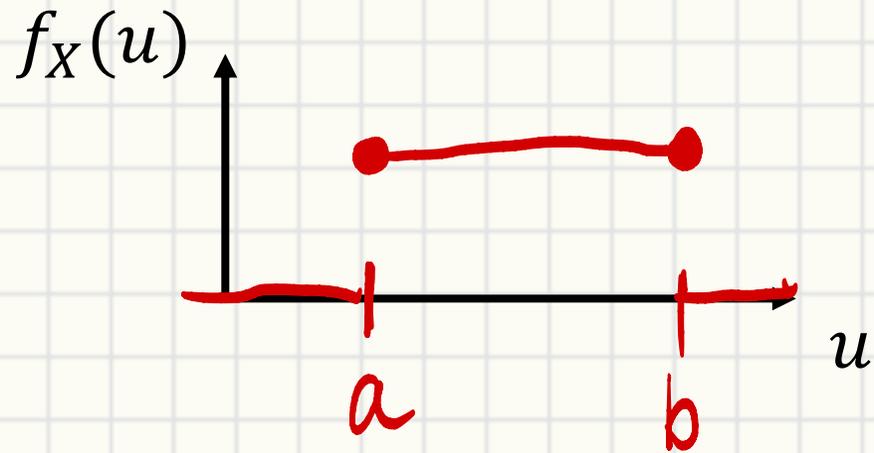
Uniform Distribution

Uniform Distribution

$$f_X(u) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \leq u \leq b \\[2mm] 0 & \text{else} \end{cases}$$

$$f_X(u) = \begin{cases} \dfrac{1}{b-a} & a \leq u \leq b \\[2mm] 0 & \text{else} \end{cases}$$

Properties

$$f_X(u) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \leq u \leq b \\ \\ 0 & \text{else} \end{cases}$$

- $E[X] = \int_{-\infty}^{\infty} u f_X(u)\, du = \displaystyle\int_a^b \frac{u}{b-a}\, du = \frac{u^2}{2}\Big|_a^b \left(\frac{1}{b-a}\right)$

$$= \frac{b^2 - a^2}{2} \times \frac{1}{b-a}$$

$$= \frac{a+b}{2}$$

- $E[X^2] = \int_{-\infty}^{\infty} u^2 f_X(u)\, du =$

  2nd momentum

- $Var(X) =$

- Special case, when $(a, b) = (0,1)$
  - k^{th} moment $E[X^k] =$
  - $Var(X) =$

$$E[X^2] = \int_a^b \frac{\mu^2}{(b-a)} \, d\mu = \frac{\mu^3}{3}\Big|_a^b \frac{1}{b-a}$$

$$= \frac{\boxed{b^3 - a^3}}{3} \times \frac{1}{b-a}$$

$$= \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

$$Var(X) = \frac{b^2 + ab + a^2}{3} - \left(\frac{a+b}{2}\right)^2 = \boxed{\frac{(a-b)^2}{12}}$$

Uniform distr. (unit uniform)

$$U \sim Uni([0, 1])$$

$$\mu_U = 0.5$$

$$\sigma_X^2 = \frac{1}{12}.$$