University of Illinois **Spring 2026**

ECE 313: Problem Set 1: Problems and Solutions

Due: Friday, January 30 at 06:59:00 p.m.
Reading: *ECE 313 Course Notes,* Chapter 1

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Defining a set of outcomes]**
   Ten balls, numbered zero through nine, are initially in a bag. Four balls are drawn out, one at a time, without replacement.

   (a) Define a sample space Ω describing the possible outcomes of this experiment. To be definite, suppose the order the four balls are drawn out is *unimportant.* Explain how the elements of your set correspond to outcomes of the experiment.

   Solution: A natural choice is $\Omega = \{\{b_1, b_2, b_3, b_4\} : 0 \leq b_i \leq 9, b_1, b_2, b_3, b_4 \text{ distinct}\}$, where for a given outcome $\{b_1, b_2, b_3, b_4\}$, b_i denotes the number on the i^{th} ball drawn from the bag.

   (b) What is the cardinality of Ω?

   Solution: $(10)(9)(8)(7)/(24) = 210$, because there are 10 possible choices for b_1, and given b_1 there are 9 possible choices for b_2, and given b_1 and b_2, there are 8 possible choices for b_3, and given b_1, b_2, and b_3, there are 7 possible choices for b_4, thus yielding $(10)(9)(8)(7) = 5040$. Finally, notice that for a given choice of $\{b_1, b_2, b_3, b_4\}$, there are $4! = 24$ permutations that correspond to the same outcome, so we need to divide 5040 by 24 which gives the desired result.

2. **[Using set theory to calculate probabilities of events]**
   Suppose A and B are two events defined on a probability space with $P(A) = 3/4$ and $P(B) = 1/2$.

   (a) If $B \subset A$, calculate $P(AB)$.

   Solution: If $B \subset A$, then $AB = B$ and $P(AB) = P(B) = 1/2$.

(b) If $A \cup B = \Omega$, calculate $P(AB)$.

Solution: Using relationship:

$$1 = P(\Omega) = P(A \cup B) = P(A) + P(B) - P(AB) = \frac{3}{4} + \frac{1}{2} - P(AB).$$

Hence we get $P(AB) = 5/4 - 1 = 1/4$.

3. **[Displaying outcomes in a two event Karnaugh map]**
Two fair dice are rolled. Let A be the event where the sum is 4 and B be the event where at least one of the numbers rolled is strictly less than 3. Suppose that order matters.

(a) Display the outcomes in a Karnaugh map.

Solution: First, consider event A. Since there are 3 possible (i.e., $4 = 1 + 3 = 2 + 2 = 3 + 1$) out of total 36 ($= 6 \times 6$) possible summations, $P(A) = 3/36 = 1/12$. For event B, one can readily see that $P(B) = 1/3 + 1/3 - 1/3 \times 1/3 = 5/9$. See below for a Karnaugh map, where $(a, b),\ a, b \in \{1, 2, 3, 4, 5, 6\}$ means that a and b are popped up after rolling the first and the second dices, respectively.

	A	A^c
B	$(2,2)$ $(1,3)$ $(3,1)$	$(1,1)$ $(1,2)$ $(1,4)$ $(1,5)$ $(1,6)$ $(2,1)$ $(2,3)$ $(2,4)$ $(2,5)$ $(2,6)$ $(3,2)$ $(4,1)$ $(4,2)$ $(5,1)$ $(5,2)$ $(6,1)$ $(6,2)$
B^c		$(3,3)$ $(3,4)$ $(3,5)$ $(3,6)$ $(4,3)$ $(4,4)$ $(4,5)$ $(4,6)$ $(5,3)$ $(5,4)$ $(5,5)$ $(5,6)$ $(6,3)$ $(6,4)$ $(6,5)$ $(6,6)$

Figure 1: Karnaugh map for Problem 3.

(b) Determine $P(AB)$.

Solution: $P(AB) = 3/36 = 1/12$.

4. **[A Karnaugh map for three events]**
There are 100 individuals in a country club that offers three sports activities: yoga, running, and zumba.

- 6 members don't participate in any sports.
- 10 members participate in all three.
- 14 members don't participate in yoga and running.
- 28 members participate in zumba and running.
- 44 members do not like yoga.
- 65 members like to run.

How many members participate in yoga but do not run?

Solution: Suppose A denotes a set of individuals participating in yoga activities; B denotes a set of individuals participating in running; and C denotes that a set of individuals participating in zumba. Then the final map is shown in the next page. Since there are 100 individuals

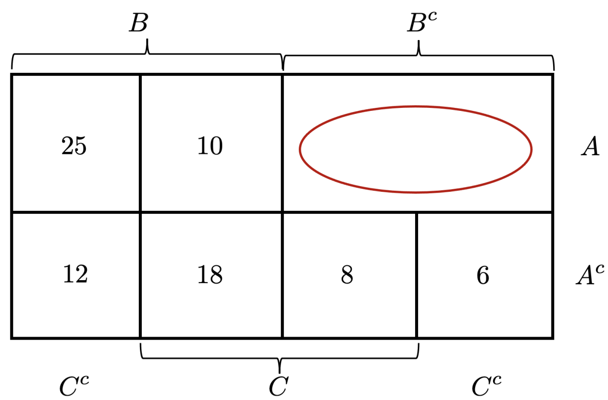

Figure 2: Karnaugh map for Problem 4.

in total, the number of members who participate in yoga but do not run, is $21 = 100 - 65 - 14$ (see red-circled part).

5. **[Selecting socks at random with a twist]**
Suppose there are eight socks in a bag which can be grouped into four pairs: $\{BR1, BR2\}$, $\{BG1, BG2\}$, $\{GR1, GR2\}$, or $\{GG1, GG2\}$. The socks of each pair have the same color (Red or Green) and are gender specific (Boy or Girl), e.g., $BR1$ is a red colored boy's socks. Suppose there are two boys and two girls present, and one at a time in a fixed order, they each draw two socks out of the bag, without replacement. Suppose all socks feel the same, so when two socks are drawn from the bag, all possibilities have equal probability. Let M be the event that each person draws a pair of socks that match in color and their gender.

    (a) Define a sample space Ω for this experiment. Suppose that the order that the people draw the socks doesn't matter–all that is recorded is which two socks each person selects.
    Solution: One choice is

$$\Omega = \big\{(A_1, A_2, A_3, A_4) : A_i \subset \{BR1, BR2, BG1, BG2, GR1, GR2, GG1, GG2\}, \\ |A_i| = 2, A_iA_j = \emptyset \text{ for all } i \neq j\big\},$$

    where an outcome $\{A_1, A_2, A_3, A_4\}$ means person i draws out the two socks in A_i ($1 \leq i \leq 4$).

    (b) Determine $|\Omega|$, the cardinality of Ω.
    Solution: $|\Omega| = 8!/2^4 = 2,520$ because there are $8!$ orders that the socks could be drawn out one at a time, but this over counts by a factor of $(2)^4$ because the order each person draws two socks doesn't matter. Another way to get this answer is to note that there are $\binom{8}{2}$ possible choices for A_1, then $\binom{6}{2}$ possible choices for A_2, then $\binom{4}{2}$ choices for A_3, then A_4 is determined as well. So $|\Omega| = \binom{8}{2}\binom{6}{2}\binom{4}{2} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3}{2\cdot2\cdot2}$

    (c) Determine the number of outcomes in M.

Solution: $|M| = 2 \cdot 2 = 4$. Notice that one specific possible outcome would be $(\{BR1, BR2\}, \{BG1, BG2\}, \{GR1, GR2\}, \{GG1, GG2\})$ in M. Other outcomes in M can be obtained if the boys exchanged their socks with each other and so did the girls leading to $2 \cdot 2 =$ permutations.

(d) Find $P(M)$.

Solution: $P(M) = \frac{|M|}{|\Omega|} = \frac{2^4 \cdot 4}{8!} = \frac{1}{7 \cdot 6 \cdot 5 \cdot 3} = \frac{1}{630}$.

(e) Find a short way to calculate $P(M)$ that doesn't require finding $|M|$ and $|\Omega|$. (Hint: Write $P(M)$ as one over an integer. Factor the integer and think about the probability of each step of the drawing procedure.)

Solution: Suppose the two boys go first followed by the two girls in drawing socks (BBGG). The first boy has a 1/2 chance of drawing the sock of the correct gender, then in the second draw has a 1/7 chance of drawing the sock which matches the color and gender of the first sock. The second boy has 1/3 chance of drawing the sock of correct gender, and in the second draw, has a 1/5 chance of drawing the sock that matches the color and gender of the first sock. The first girl draws a sock and then has a 1/3 chance of drawing a sock of the right color. The remaining two socks are left for the second girl. So $P(M) = (1/2)(1/7)(1/3)(1/5)(1/3) = \frac{1}{630}$.

If instead the boys and girls took turns, i.e., first boy, first girl, second boy, and the second girl, to draw (BGBG), then the first boy has a 1/2 chance of drawing the sock of the correct gender, then in the second draw has a 1/7 chance of drawing the sock which matches the color and gender of the first sock. The first girl has 2/3 chance of drawing the sock of correct gender, and in the second draw, has a 1/5 chance of drawing the sock that matches the color and gender of the first sock. The second boy has a 1/2 chance of drawing a sock of the right gender and in the second draw a 1/3 chance of getting the matching sock. The remaining two socks are left for the second girl. Thus, $P(M) = (1/2)(1/7)(2/3)(1/5)(1/2)(1/3) = \frac{1}{630}$.

For GGBB, it follows the same computational procedure as the BBGG case. For GBGB, BGGB and GBBG, they follow the same computational procedure as the BGBG case. Therefore, it does not matter in what sequence the individuals take turns to draw.

6. **[Two more poker hands]**

Suppose five cards are drawn from a standard 52 card deck of playing cards, as described in Example 1.4.3, with all possibilities being equally likely.

(a) *FLUSH* is the event that all five cards have the same suit. Find $P(\mathit{FLUSH})$.

Solution: There are $\binom{13}{5}$ ways to select the numbers for the five cards, then 4 ways to choose the suit. Thus,

$$P(\mathit{FLUSH}) = \frac{\binom{13}{5} \cdot 4}{\binom{52}{5}} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 4}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \approx 0.00198$$

(b) *TWO PAIRS* is the event there are two different pairs in the five cards. Find $P(\mathit{TWO\ PAIRS})$. (Hint: *FULL HOUSE* and *FOUR OF A KIND* should not count as TWO PAIRS)

Solution: There are $\binom{13}{2}$ ways to select the numbers for the two pairs, then 11 ways to choose the number on the fifth card, then $\binom{4}{2} = 6$ ways to choose suits for the cards in one pair, 6 ways to choose suits for the cards of the other pair, and 4 ways to choose the suit of the unpaired card. Thus,

$$\begin{aligned} P(\mathit{TWO\ PAIR}) &= \frac{\binom{13}{2}(11)(6)^2 4}{\binom{52}{5}} \\ &= \frac{3 \cdot 6 \cdot 11}{5 \cdot 17 \cdot 49} \approx 0.0475 \end{aligned}$$

7. **[Fishing]**
Harry's backyard pond contains 6 goldfish and 4 catfish. All fish are equally likely to be caught.

(a) Suppose that Harry catches a total of 5 fish (no fish are thrown back). Let G be the event that Harry catches exactly 3 goldfish. What is the probability $P(G)$?
Solution: There are $\binom{10}{5}$ ways to catch 5 fish. Of these, there are $\binom{6}{3} \cdot \binom{4}{2}$ ways of catching 3 goldfish and 2 catfish. Thus,

$$\begin{aligned} P(G) &= \frac{\binom{6}{3} \cdot \binom{4}{2}}{\binom{10}{5}} \\ &= \frac{10}{21} \approx 0.476. \end{aligned}$$

(b) Assume event G occurs in the first catch and suppose that all 5 fish are returned to the pond. Harry starts fishing again. This time he catches a total of 3 fish. Let A be the event that among the caught set of 3 fish, exactly 2 goldfish are included that were also caught in the first catch. (Assume that fish do not learn from experience.) Find $P(A)$.
Solution: There are $\binom{10}{3}$ ways to catch 3 fish. Of these, there are $\binom{3}{2}$ ways of catching exactly two goldfish that were previously caught. Notice that there are $\binom{7}{1}$ ways of catching the third fish since the third fish should not be the remaining one goldfish caught in the first place. This gives:

$$\begin{aligned} P(A) &= \frac{\binom{3}{2} \cdot \binom{7}{1}}{\binom{10}{3}} \\ &= \frac{7}{40} = 0.175. \end{aligned}$$