University of Illinois **Spring 2025**

ECE 313: Problem Set 10: Problems and Solutions

Due: Friday, April 18 at 07:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.2 - 4.3

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. **Please write down your work and derivations. An answer without justification as of how it is found will not be accepted.** You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME AS IT APPEARS ON CANVAS
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PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. [**Joint pmf I**]
   The joint pmf $p_{X,Y}(u,v)$ of X and Y is shown in the table below:

	$u=0$	$u=1$	$u=2$	$u=3$
$v=4$	0	0.1	0.1	0.2
$v=5$	0.2	0	0	0
$v=6$	0	0.2	0.1	0.1

   (a) Find the marginal pmfs $p_X(u)$ and $p_Y(v)$.
   Solution: The marginal $p_X(u)$ can be evaluated as the sum $\sum_{k=4}^{6} p_{X,Y}(u,k)$. Therefore

$$p_X(u) = \begin{cases} 0.2, & u=0, \\ 0.3, & u=1 \\ 0.2, & u=2 \\ 0.3 & u=3 \\ 0 & \text{otherwise} \end{cases}$$

Following the similar definition for $p_Y(v)$, we get

$$p_Y(v) = \begin{cases} 0.4, & v = 4, \\ 0.2, & v = 5 \\ 0.4, & v = 6 \\ 0 & \text{otherwise} \end{cases}$$

(b) Let $Z = X + Y$. Find p_Z, the pmf of Z.
Solution: This can be counted. Z can only be realized as an integer between 4 and 9, and by analyzing each (X, Y) pair, we have

$$p_Z(z) = \begin{cases} p_{X,Y}(0,4) & z = 4, \\ p_{X,Y}(1,4) + p_{X,Y}(0,5) & z = 5 \\ p_{X,Y}(2,4) + p_{X,Y}(1,5) + p_{X,Y}(0,6) & z = 6 \\ p_{X,Y}(3,4) + p_{X,Y}(2,5) + p_{X,Y}(1,6) & z = 7 \\ p_{X,Y}(3,5) + p_{X,Y}(2,6) & z = 8 \\ p_{X,Y}(3,6) & z = 9 \\ 0 & \text{otherwise} \end{cases}$$

Simplifying gives

$$p_Z(z) = \begin{cases} 0.3 & z = 5 \\ 0.1 & z = 6 \\ 0.4 & z = 7 \\ 0.1 & z = 8 \\ 0.1 & z = 9 \\ 0 & \text{otherwise} \end{cases}$$

(c) Find $p_{Y|X}(v|3)$ for all v and find $E[Y|X = 3]$.
Solution: By definition, we know that

$$p_{Y|X}(v|3) = \frac{p_{X,Y}(3,v)}{p_X(3)}, \quad v = 4, 5, 6$$

We've already computed the marginal probability in part (a), so we can just plug in the values from the table.

$$p_{Y|X}(v|3) = \begin{cases} \frac{2}{3} & v = 4 \\ \frac{1}{3} & v = 6 \\ 0 & \text{otherwise} \end{cases}$$

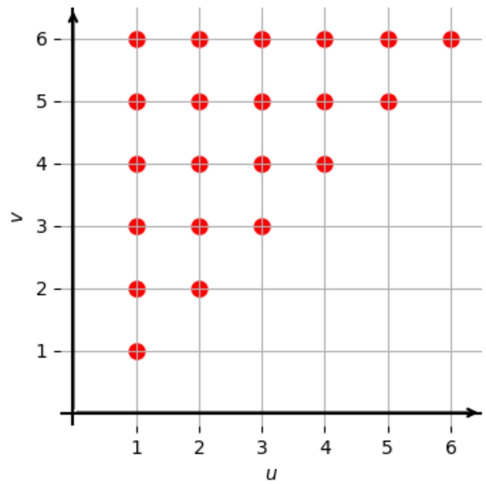
Then we can follow the definition of expected value.

$$\begin{aligned} E[Y|X = 3] &= \sum_{k=4}^{6} k \cdot p_{Y|X}(k|3) \\ &= 4 \cdot \frac{2}{3} + 6 \cdot \frac{1}{3} \\ &= \frac{14}{3} \end{aligned}$$

2. **[Joint pmf II]**
Two fair six-sided dice are rolled. One of the dice shows Z_1 pips (pips are the small dots on each face of a six-sided die), the other shows Z_2 pips. The random variables X and Y are defined as follows:

$$X = \min(Z_1, Z_2)$$
$$Y = \max(Z_1, Z_2)$$

(a) Sketch the support, in the (u, v) plane, of the joint pmf $p_{X,Y}(u, v)$.
Solution: The support is the set of all pairs of integers (u, v) such that $1 \le u \le v \le 6$.



(b) Find the joint pmf $p_{X,Y}(u, v)$.
Solution: If $u \neq v$, then there are two ways in which $X = u$, $Y = v$, can take values, either $Z_1 = u$ and $Z_2 = v$ or vice versa. On the other hand, if $u = v$, then there is only one way in which $X = u$, $Y = v$ can occur. Thus,

$$p_{X,Y}(u, v) = \begin{cases} \frac{1}{18}, & 1 \le u < v \le 6; \\ \frac{1}{36}, & 1 \le u = v \le 6; \\ 0, & \text{otherwise.} \end{cases}$$

(c) Find the marginal pmf of Y.
Solution: There are $v - 1$ ways in which $X = u$, $Y = v$ and $u < v$ can occur. There is one way in which $X = Y = u = v$ can occur. Thus,

$$p_Y(v) = \begin{cases} \frac{1}{18}(v-1) + \frac{1}{36} = \frac{2v-1}{36}, & 1 \le v \le 6; \\ 0, & \text{otherwise.} \end{cases}$$

(d) Find $E[Y - X]$.
Solution: The value $(Y - X) = (v - u)$ is a constant along every diagonal of the (u, v) plane. There are five points for which $(v - u) = 1$, four for which $(v - u) = 2$, and so on, thus

$$E[Y - X] = \sum_{u,v} (v - u) p_{X,Y}(u, v) = \frac{1 \times 5}{18} + \frac{2 \times 4}{18} + \frac{3 \times 3}{18} + \frac{4 \times 2}{18} + \frac{5 \times 1}{18} = \frac{35}{18}.$$

Alternatively, you can find $E[Y]-E[X]$ instead due to the linearity of expectation.

$$E[Y]=1\cdot\frac{1}{36}+2\cdot\frac{3}{36}+3\cdot\frac{5}{36}+4\cdot\frac{7}{36}+5\cdot\frac{9}{36}+6\cdot\frac{11}{36}=\frac{161}{36}$$
$$E[X]=6\cdot\frac{1}{36}+5\cdot\frac{3}{36}+4\cdot\frac{5}{36}+3\cdot\frac{7}{36}+2\cdot\frac{9}{36}+1\cdot\frac{11}{36}=\frac{91}{36}$$
$$E[Y]-E[X]=\frac{161-91}{36}=\frac{35}{18}$$

3. **[Joint pdf I]**
The jointly continuous random variables X and Y have joint pdf

$$f_{X,Y}(u,v)=\begin{cases}1.5 & 0\le u<1,\ 0\le v<1,\ 0\le u+v<1,\\ 0.5 & 0\le u<1,\ 0\le v<1,\ 1\le u+v<2,\end{cases}$$

and zero elsewhere.

(a) Find the marginal pdf of Y.
Solution:
By definition, we know that $f_Y(v)=\int_{\mathbb{R}}f_{X,Y}(u,v)du$. Now we need to determine the relevant values of u to integrate over where $f_{X,Y}(u,v)$ takes a non-zero value.
For the first case, we can see that $u\in[0,1)$ and $u\in[-v,1-v)$. Since we know that our joint pdf is only nonzero when $v\in[0,1)$, we find that the intersection of the two sets becomes $u\in[0,1-v)$.
Similarly, for the second case, we find that $u\in[1-v,1)$.
Therefore, when $v\in[0,1)$

$$\begin{aligned}f_Y(v)&=\int_{\mathbb{R}}f_{X,Y}(u,v)du\\&=\int_0^{1-v}1.5du+\int_{1-v}^{1}0.5du\\&=1.5-1.5v+0.5v\\&=1.5-v\end{aligned}$$

So

$$f_Y(v)=\begin{cases}1.5-v & 0\le v<1\\ 0 & \text{otherwise}\end{cases}$$

(b) Find $P(X+Y\ge\frac{3}{2})$.
Solution:
If $X+Y\ge\frac{3}{2}$, then for all realizations of X and Y, $X\in[\frac{1}{2},1)$. Let X be realized as u. Then $Y\in[\frac{3}{2}-u,1)$. We also know that for any realization (u,v) over these sets, $f_{X,Y}(u,v)=0.5$. Integrating over the set of relevant values gives us

$$\begin{aligned}P(X+Y\ge\frac{3}{2})&=\int_{\frac{1}{2}}^{1}\int_{\frac{3}{2}-u}^{1}f_{X,Y}(u,v)dvdu\\&=\frac{1}{2}\int_{\frac{1}{2}}^{1}(u-0.5)du\\&=\frac{1}{16}\end{aligned}$$

Alternatively, considering the geometric interpretation, the double integral can be regarded as the volume under the function in the required region, which is bounded by $u < 1$, $v < 1$, and $u + v \geq \frac{3}{2}$. This gives

$$P(X + Y \geq \frac{3}{2}) = f_{X,Y}(u, v) \cdot \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right) = \frac{1}{16}$$

(c) Find $P(X^2 + Y^2 \leq 1)$.

Solution:
If $X^2 + Y^2 \leq 1$, then for all realizations of X and Y, $X \in [0, 1)$. Let X be realized as u. Then $Y \in [0, \sqrt{1 - u^2})$. Therefore we know that

$$P(X^2 + Y^2 \leq 1) = \int_0^1 \int_0^{\sqrt{1-u^2}} f_{X,Y}(u, v) dv du$$

We can see that for any realization of X and Y, if $X+Y \leq 1$, then $X^2+Y^2 \leq 1$. However, the reverse is not always true – there exist realizations of X, Y such that $X^2 + Y^2 \leq 1$ but $X + Y > 1$ e.g. $X = Y = \frac{3}{5}$. We can quickly see that $1 - u \leq 1 - u^2 \leq \sqrt{1 - u^2}$ for $0 \leq u \leq 1$. Therefore

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-u^2}} f_{X,Y}(u, v) dv du &= \int_0^1 \left(\int_0^{1-u} 1.5 dv + \int_{1-u}^{\sqrt{1-u^2}} 0.5 dv\right) du \\
&= \int_0^1 \frac{3}{2} - \frac{3}{2}u + \frac{1}{2}\left(\sqrt{1 - u^2} - (1 - u)\right) du \\
&= \int_0^1 -u + \frac{1}{2}\sqrt{1 - u^2} + 1 du \\
&= \frac{1}{2}\int_0^1 \sqrt{1 - u^2} du + \frac{1}{2}
\end{aligned}$$

We can use a trig substitution such that $u = \sin(x)$ and $du = \cos(x)dx$. Therefore

$$\begin{aligned}
&= \frac{1}{2}\int_0^{\pi/2} \sqrt{1 - \sin(x)^2}\cos(x)dx + \frac{1}{2} \\
&= \frac{1}{2}\int_0^{\pi/2} \cos(x)^2 dx + \frac{1}{2} \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{1 + \cos(2x)}{2} dx + \frac{1}{2} \\
&= \frac{1}{4}\left(x + \frac{1}{2}\sin(2x)\right)\Big|_0^{\pi/2} + \frac{1}{2} \\
&= \frac{\pi}{8} + \frac{1}{2}
\end{aligned}$$

Note: You may notice that $\int_0^1 \sqrt{1 - u^2} du$ denotes the area of the intersection of Unit circle and Quadrant I, which is $\pi/4$ as well.

Alternatively, you may also compute as

$$
\begin{aligned}
P(X^2+Y^2 \le 1) &= 1 - P(X^2+Y^2 > 1) \\
&= 1 - \int_0^1 \int_{\sqrt{1-u^2}}^1 0.5 dv du \\
&= 1 - \frac{1}{2}\int_0^1 (1-\sqrt{1-u^2}) du \\
&= 1 - \left(\frac{1}{2} - \frac{\pi}{8}\right) \\
&= \frac{\pi}{8} + \frac{1}{2}
\end{aligned}
$$

or using geometric interpretation as in (b):

$$
\begin{aligned}
P(X^2+Y^2 \le 1) &= 1.5 \cdot \frac{1}{2} + 0.5 \cdot \left(\frac{\pi}{4} - \frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{\pi}{8}
\end{aligned}
$$

4. **[Joint pdf II]**
X and Y are two random variables with the following joint pdf:

$$
f_{X,Y}(u,v) = \begin{cases} A(1-|u-v|), & 0<u<1, 0<v<1; \\ 0, & \text{otherwise.} \end{cases}
$$

(a) Find A.
Solution: We get A using the fact that the integral of the pdf over the entire region has to be 1. To this end, we first divide the region that (X,Y) can take, depending on the values of u and v:

$$
f_{X,Y}(u,v) = \begin{cases} A(1-(u-v)), & u \ge v; \\ A(1+(u-v)), & u < v. \end{cases}
$$

Thus,

$$
\int_0^1 \int_0^u A(1-u+v) dv du + \int_0^1 \int_u^1 A(1+u-v) dv du = 1 \iff A = 3/2.
$$

(b) Find marginal pdfs for X and Y.
Solution: The support of f_X is the interval $[0,1]$. For $0 \le u \le 1$,

$$
f_X(u) = \int_{\mathbb{R}} f(u,v) dv = \int_0^u A(1-u+v) dv + \int_u^1 A(1+u-v) dv.
$$

Therefore,

$$
f_X(u) = \begin{cases} \frac{-3u^2}{2} + \frac{3u}{2} + \frac{3}{4}, & 0<u<1; \\ 0, & \text{else.} \end{cases}
$$

Notice that by symmetry, $f_{X,Y}(u,v) = f_{X,Y}(v,u)$, or in other words, (X,Y) has the same joint pdf as (Y,X). As a result, the two marginal pdfs f_X and f_Y are identical.

(c) Find $P\{X > Y\}$.

Solution: By symmetry, $P\{X > Y\} = 1/2$.

Alternatively, you may also find this probability using the general method of integration. With $X > Y$, we get:

$$\begin{aligned} P\{X > Y\} &= \int_0^1 \int_0^u A(1-u+v)dvdu \\ &= \frac{3}{2}\int_0^1 \int_0^u (1-u+v)dvdu \\ &= \frac{3}{2}\int_0^1 (u - \frac{1}{2}u^2)du \\ &= \frac{3}{2}\left(\frac{1}{2} - \frac{1}{6}\right) \\ &= \frac{1}{2} \end{aligned}$$

(d) Find $P\{X + Y < 1 | X > 1/2\}$.

Solution: First, use the pdf of X to compute $P\{X > \frac{1}{2}\} = \int_{0.5}^1 f_X(u)du = 0.5$. Also, $P\{X+Y < 1, X > 1/2\}$ is the integral of the joint pdf over the shaded region (as shown in Fig.1).

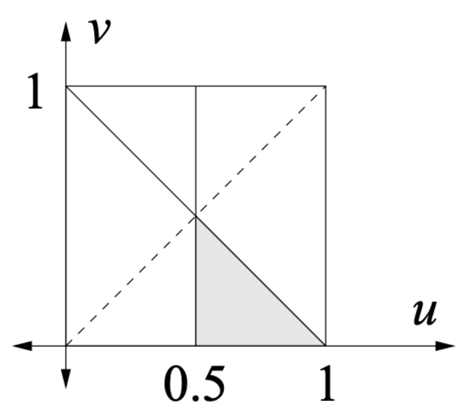


Figure 1: Region of support for Subpart (d).

Then we have $P\{X + Y < 1, X > 1/2\} = \int_{0.5}^1 \int_0^{1-u} \frac{3}{2}(1-u+v)dvdu = \frac{3}{32}$, where $f_{X,Y}(u,v)$ is defined to be $\frac{3}{2}(1-u+v)$ over the integral because $X+Y < 1$, and $X > \frac{1}{2}$, meaning that $\frac{1}{2} + Y < 1$ and therefore $Y < \frac{1}{2}$.

Finally,

$$P\{X + Y < 1 | X > 1/2\} = \frac{P\{X+Y < 1, X > 1/2\}}{P\{X > 1/2\}} = \frac{3}{16}.$$