ECE 313: Problem Set 9: Problems and Solutions

Due: Sunday, April 13 at 11:59 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.8, 3.10, 4.1

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. **This homework is instead due by 11:59 p.m. on the following Sunday. Please write down your work and derivations. An answer without justification as of how it is found will not be accepted.** You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.
Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Function of a RV 1]**
   Let X be a uniform random variable with support $[-1, 2]$ and let $Y = X + |X|$. Find the CDF of Y.

   Solution: In order to get the pdf of Y, we can first determine its support. Depending on X, Y can equal two things:

$$Y = \begin{cases} X - X & X < 0 \\ X + X & X \geq 0 \end{cases}$$

   or

$$Y = \begin{cases} 0 & X < 0 \\ 2X & X \geq 0 \end{cases}$$

   The support of Y is therefore $[0, 4]$. We can also note that Y is a mixed random variable, since Y is equal to 0 for outcomes of non-zero probability.

   For $c = 0$,

$$\begin{aligned} P(Y \leq c) &= P(Y < 0) + P(Y = 0) \\ &= P(Y = 0) \\ &= P(-1 \leq X < 0) \\ &= \frac{1}{3} \end{aligned}$$

and for $0 < c \le 4$,

$$\begin{aligned} P(Y \le c) &= P(0 < Y \le c) + P(Y = 0) \\ &= P(0 < Y \le c) + \frac{1}{3} \\ &= P(0 < 2X \le c) + \frac{1}{3} \\ &= P(0 < X \le \frac{c}{2}) + \frac{1}{3} \\ &= \frac{c+2}{6} \end{aligned}$$

As a result, the CDF of Y is

$$F_Y(c) = \begin{cases} 0 & c < 0 \\ \frac{1}{3} & c = 0 \\ \frac{c+2}{6} & 0 < c \le 4 \\ 1 & c > 4 \end{cases}$$

2. **[Function of a RV 2]**
   Let X be a exponential random variable with parameter $\lambda = 1$. Let $Y = 1/X$. Find the pdf of Y.

   Solution: The support of Y is $(0, \infty)$. Consider the CDF of Y, for $v > 0$,

$$\begin{aligned} F_Y(v) &:= \Pr(Y \le v) = \Pr\left(\frac{1}{X} \le v\right) = \Pr\left(X \ge \frac{1}{v}\right) \\ &= 1 - F_X\left(\frac{1}{v}\right) \\ &= 1 - (1 - e^{-\lambda/v}) \\ &= e^{-\lambda/v} \end{aligned}$$

   and $F_Y(v) = 0$ otherwise. Differentiating $F_Y(v)$ w.r.t. v and finally plugging in $\lambda = 1$, we get the following pdf:

$$\begin{aligned} f_Y(v) &= \frac{\partial}{\partial v} F_Y(v) = \frac{\lambda}{v^2} \cdot e^{-\lambda/v} \\ &= \frac{e^{-1/v}}{v^2}, \quad \text{for } v > 0 \end{aligned}$$

   and $f_Y(v) = 0$ otherwise.

3. **[Generating a Weibull Distribution]**
   Let X be uniformly distributed on $(0, 1)$, and let $Y = g(X)$, where $g(\cdot)$ is a function of X. We want Y to have the CDF of a Weibull distribution with shape parameter $\beta > 0$ and scale parameter $\alpha > 0$; that is, $F_Y(v) = 1 - e^{-\left(\frac{v}{\alpha}\right)^\beta}$ for $v \ge 0$, and zero otherwise. Find a function $g(\cdot)$ to accomplish this, and check that this indeed gives the desired distribution.

   Solution: As indicated in Section 3.8.2, an appropriate function $g(\cdot)$ is given by the inverse function of $F_Y(v)$. To find the inverse, let

$$s = F_Y(v) = 1 - e^{-(v/\alpha)^\beta},$$

such that

$$v = \alpha(-\ln(1-s))^{1/\beta}.$$

This gives $Y = g(U) = \alpha(-\ln(1-U))^{1/\beta}$, where U denotes a uniform distribution random variable over the interval $[0,1]$. Together with $\alpha, \beta > 0$ (as mentioned in the problem statement), one can readily check whether or not this indeed gives the desired distribution as follows:

$$\begin{aligned}
F_Y(v) :=& \Pr(Y \le v) = \Pr(g(U) \le v) \\
=& \Pr(\alpha(-\ln(1-U))^{1/\beta} \le v) \\
=& \Pr(U \le 1 - e^{-(v/\alpha)^\beta}) \\
=& F_U(1 - e^{-(v/\alpha)^\beta}) \\
=& 1 - e^{-(v/\alpha)^\beta}.
\end{aligned}$$

4. **[Binary Hypothesis Testing 1]**
   Consider the following binary hypothesis testing problem. Under H_0, the random variable X has the pdf f_0, while under H_1, the random variable X has the pdf f_1, where

$$f_0(u) = \begin{cases} \frac{1}{4} & u \in \left[\frac{-1}{2}, \frac{3}{2}\right] \cup \left[\frac{5}{2}, \frac{9}{2}\right], \\ 0 & \text{else} \end{cases}$$

   and

$$f_1(u) = \begin{cases} \frac{1}{4}u & u \in [0,2], \\ \frac{-1}{4}u + 1 & u \in (2,4], \\ 0 & \text{else} \end{cases}$$

   Assume that $4\pi_0 = \pi_1$.

   (a) Find the ML rule.
   Solution: One can compute the likelihood ratio to describe the ML rule:

$$\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \begin{cases} 0, & u \in [-\frac{1}{2}, 0] \cup [4, \frac{9}{2}]; \\ u, & u \in [0, \frac{3}{2}]; \\ \infty, & u \in [\frac{3}{2}, \frac{5}{2}]; \\ 4-u, & u \in [\frac{5}{2}, 4]. \end{cases}$$

   The ML rule declares H_1 if $\Lambda(u) \ge 1$. Hence the ML rule declares H_1 if $X \in [1,3]$, and declares H_0 otherwise. Another simple way of getting the ML rule is by plotting both pdfs on the same graph and declaring H_1 whenever $f_1(u) \ge f_0(u)$.

   (b) Find $p_{false\,alarm}$, p_{miss}, and p_e for the ML rule.
   Solution:

$$p_{\text{false-alarm}} := \Pr(\text{declare } H_1 | H_0 \text{ true}) = \Pr(X \in [1,3] | H_0 \text{ true}) = \frac{1}{4}\frac{1}{2} + \frac{1}{4}\frac{1}{2} = \frac{1}{4},$$

   where the second-to-last equality corresponds to the probability that X lies in $[1,3]$ out of the total intervals that X can be, under f_0. Furthermore,

$$\begin{aligned}
p_{\text{miss}} :=& \Pr(\text{declare } H_0 | H_1 \text{ true}) \\
=& \Pr(X \notin [1,3] | H_1 \text{ true}) = \int_0^1 \frac{1}{4}u du + \int_3^4 \left(1 - \frac{1}{4}u\right) du = \frac{1}{4},
\end{aligned}$$

where the second last equality corresponds to the probability that X lies not in $[1,3]$ out of the total intervals that X can be, under f_1.

And the error probability is $p_e = \pi_0 p_{\text{false-alarm}} + \pi_1 p_{\text{miss}} = \frac{1}{4}(\pi_0 + \pi_1) = \frac{1}{4}$.

(c) Find the MAP rule.

Solution: One can use the likelihood ration test again, where now the MAP rule declares H_1 if $\Lambda(u) \geq \frac{\pi_0}{\pi_1} = \frac{1}{4}$, and declares H_0 otherwise.

Another simple way of getting the solution is by plotting f_1 and $\frac{\pi_0}{\pi_1} f_0 = f_0/4$ and declaring H_1 whenever $f_1(u) \geq f_0(u)/4$.

(d) Find $p_{false\, alarm}$, p_{miss}, and p_e for the MAP rule.

Solution:

$$p_{\text{false-alarm}} := \Pr(\text{declare } H_1 | H_0 \text{ true}) = \Pr\left(X \in \left[\frac{1}{4}, \frac{15}{4}\right] | H_0 \text{ true}\right) = \frac{1}{4}\frac{5}{4} + \frac{1}{4}\frac{5}{4} = \frac{5}{8},$$

where the second last equality corresponds to the probability that X lies in $[\frac{1}{4}, \frac{15}{4}]$ out of the total intervals that X can be at, under f_0. Furthermore,

$$\begin{aligned} p_{\text{miss}} :=& \Pr(\text{declare } H_0 | H_1 \text{ true}) \\ =& \Pr\left(X \notin \left[\frac{1}{4}, \frac{15}{4}\right] | H_1 \text{ true}\right) = \int_0^{\frac{1}{4}} \frac{1}{4} u du + \int_{\frac{15}{4}}^{4} \left(1 - \frac{1}{4} u\right) du = \frac{1}{8}\frac{1}{16} + \frac{1}{8}\frac{1}{16} = \frac{1}{64}, \end{aligned}$$

where the second last equality corresponds to the probability that X lies not in $[\frac{1}{4}, \frac{15}{4}]$ out of the total intervals that X can be at, under f_1.

And the error probability is $p_e = \pi_0 p_{\text{false-alarm}} + \pi_1 p_{\text{miss}} = \frac{1}{5}\frac{5}{8} + \frac{4}{5}\frac{1}{64} = \frac{11}{80}$.

5. **[Binary Hypothesis Testing 2]**

Consider a binary hypothesis testing problem where the observation X is exponentially distributed with parameter λ under H_0 and uniformly distributed in $[0, b]$ under H_1.

(a) Find the value(s) of π_0 (the prior probability H_0 is true) such that the MAP decision rule would always select H_0.

Solution: According to the MAP decision rule, we declare H_0 if

$$\Lambda(x) = \frac{p_1}{p_0} < \frac{\pi_0}{\pi_1}.$$

For $X > b$, MAP rule will always select H_0 because p_1 corresponding to the uniform distribution is zero. For $0 \leq X \leq b$, MAP rule will declare H_0 if

$$\begin{aligned} \frac{1/b}{\lambda e^{-\lambda x}} &< \frac{\pi_0}{1 - \pi_0} \\ \frac{1}{1 + b\lambda e^{-\lambda x}} &< \pi_0 \end{aligned}$$

For $0 \leq X \leq b$, the maximum value of $\frac{1}{1+b\lambda e^{-\lambda x}}$ is $\frac{1}{1+b\lambda e^{-\lambda b}}$. Besides, π_0 is a probability so it is upper bounded by 1. Therefore, the necessary and sufficient condition on π_0 is

$$\frac{1}{1 + b\lambda e^{-\lambda b}} < \pi_0 \leq 1$$

(Note: using two $\leq$ signs is also accepted.)

(b) Find the probability of error p_e for the MAP decision rule under the above condition.
Solution: The probability of error p_e is

$$\begin{aligned} p_e &= \pi_0 P(H_1|H_0) + \pi_1 P(H_0|H_1) \\ &= \pi_1 P(H_0|H_1) \\ &= 1 - \pi_0 \end{aligned}$$

We used the fact that for the above values of π_0, the MAP rule will always select H_0, i.e. $P(H_0|H_1) = 1$.

6. **[Joint CDFs]**
Consider the following function:

$$F(u, v) = \begin{cases} 0, & u + v \leq 1, \\ 1, & u + v > 1. \end{cases}$$

Is this a valid joint CDF? Why or why not? Prove your answer and show your work.

Solution: This CDF is not right-continuous in either u or v, and therefore it is not a valid joint CDF. You may also give a set of values for (a, b, c, d) satisfying $a < b$ and $c < d$ such that $F(b, d) - F(b, c) - F(a, d) + F(a, c) < 0$, e.g. $(a, b, c, d) = (0, 2, 0, 2)$.