University of Illinois **Spring 2025**

ECE 313: Problem Set 4: Problems and Solutions

Due: Friday, February 28 at 7 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.6 - 2.9.

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME
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SECTION
PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. [**Bumped From The Flight**]
   Suppose that 105 passengers hold reservations for a 100-passenger flight. The number of passengers who show up at the gate can be modeled as a binomial random variable $X \sim Binomial(105, 0.9)$.

   (a) On average, how many passengers show up at the gate?
   Solution: The mean of a Binomial distribution with $n = 105$ trails and probability of success $p = 0.9$ is

   $$E[X] = np = 94.5$$

   (b) If $X \leq 100$, everyone who shows up gets to board. Find $P(X \leq 100)$.
   Solution:

   $$\begin{aligned} P(X \leq 100) &= 1 - P(X > 100) \\ &= 1 - \sum_{k=101}^{105} P(X = k) \\ &= 1 - \sum_{k=101}^{105} \binom{105}{k} 0.9^k (0.1)^{105-k} \\ &\approx 0.98328368 \end{aligned}$$

   (c) Explain why the number of no-shows can be modeled as a binomial random variable $Y \sim Binomial(105, 0.1)$.

Solution: The number of no-shows $Y = y$ is distributed as

$$\begin{aligned} P(Y=y) &= P(X = 105 - y) \\ &= \binom{105}{105-y} 0.9^{105-y} 0.1^{105-(105-y)} \\ &= \binom{105}{y} 0.1^{y} 0.9^{105-y} \end{aligned}$$

where we used

$$\binom{105}{105-y} = \binom{105}{y}$$

Hence, $Y \sim Binomial(105, 0.1)$.

(d) Notice that the probability that everyone who shows up gets to board can also be expressed as $P(Y \geq 5)$. Use the Poisson approximation to compute $P(Y \geq 5)$ and compare your answer to the exact answer that you found in part (b).

Solution: We can use the Poisson approximation for a Binomial distribution when n is large, p is small and $\lambda = np$. For the Binomially distributed random variable Y, we can approximate it by using a Poisson distribution with $\lambda = 105 \cdot 0.1 = 10.5$,

$$\begin{aligned} P(Y \geq 5) &= 1 - P(Y < 5) \\ &= 1 - \sum_{k=0}^{4} P(Y = k) \\ &= 1 - \sum_{k=0}^{4} \frac{\lambda^k e^{-\lambda}}{k!} \\ &= 0.97890643 \end{aligned}$$

The Poisson approximation is very close to the exact value and differs by less than 0.5%

2. **[Binomial and Poisson]**

Solve the following questions.

(a) Let $X \sim \text{Bin}(n, p)$. Find $P(X \text{ is odd})$ in terms of n and p.

Solution: Let $q = 1 - p$. We can use the binomial series to help us solve $P(X \text{ is even}) + P(X \text{ is odd})$ and $P(X \text{ is even}) - P(X \text{ is odd})$. We have

$$\begin{aligned} &P(X \text{ is even}) + P(X \text{ is odd}) \\ &= \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \binom{n}{3} p^3 q^{n-3} + \dots \\ &= (q+p)^n = 1 \end{aligned}$$

and

$$\begin{aligned} &P(X \text{ is even}) - P(X \text{ is odd}) \\ &= \binom{n}{0} p^0 q^n - \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} - \binom{n}{3} p^3 q^{n-3} + \dots \\ &= \binom{n}{0} (-p)^0 q^n + \binom{n}{1} (-p)^1 q^{n-1} + \binom{n}{2} (-p)^2 q^{n-2} + \binom{n}{3} (-p)^3 q^{n-3} + \dots \\ &= (q-p)^n \end{aligned}$$

Therefore,

$$P(X \text{ is odd}) = \frac{(q+p)^n - (q-p)^n}{2} = \frac{1-(1-2p)^n}{2}$$

(b) Let $Y \sim \text{Poi}(\lambda)$. Find $P(Y \text{ is odd})$ in terms of λ.
Solution: We have

$$\begin{aligned} P(Y \text{ is even}) + P(Y \text{ is odd}) &= e^{-\lambda}\frac{\lambda^0}{0!} + e^{-\lambda}\frac{\lambda^1}{1!} + e^{-\lambda}\frac{\lambda^2}{2!} + e^{-\lambda}\frac{\lambda^3}{3!} + \dots \\ &= e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} \\ &= e^{-\lambda}e^{\lambda} = 1 \end{aligned}$$

and

$$\begin{aligned} P(Y \text{ is even}) - P(Y \text{ is odd}) &= e^{-\lambda}\frac{\lambda^0}{0!} - e^{-\lambda}\frac{\lambda^1}{1!} + e^{-\lambda}\frac{\lambda^2}{2!} - e^{-\lambda}\frac{\lambda^3}{3!} + \dots \\ &= e^{-\lambda}\frac{(-\lambda)^0}{0!} + e^{-\lambda}\frac{(-\lambda)^1}{1!} + e^{-\lambda}\frac{(-\lambda)^2}{2!} + e^{-\lambda}\frac{(-\lambda)^3}{3!} + \dots \\ &= e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!} \\ &= e^{-\lambda}e^{-\lambda} = e^{-2\lambda} \end{aligned}$$

Therefore,

$$P(Y \text{ is odd}) = \frac{1-e^{-2\lambda}}{2}$$

(c) Suppose that $n \to \infty$ and $p \to 0$ such that $np = \lambda$. Verify that your answer in part (a) converges to the answer in part (b). (Hint: $(1-\frac{\lambda}{n})^n \to e^{-\lambda}$ as $n \to \infty$).
Solution: As stated in (2.9) in the course notes,

$$(1-\frac{\lambda}{n})^n \to e^{-\lambda} \text{ as } n \to \infty$$

Therefore,

$$\begin{aligned} P(X \text{ is odd}) &= \frac{1-(1-2p)^n}{2} \\ &= \frac{1-(1-\frac{2\lambda}{n})^n}{2} \\ &\to \frac{1-e^{-2\lambda}}{2} \\ &= P(Y \text{ is odd}) \text{ as } n \to \infty \end{aligned}$$

3. **[Maximum Likelihood Parameter Estimation]**
A biased coin when tossed shows a Heads with probability p and Tails with probability $1-p$.

(a) The biased coin is tossed 10 times and 6 Heads are observed. What is the maximum likelihood estimate $\hat{p}_{ML}$ of p given this observation?

Solution: The likelihood function is

$$L(p) = P\{6 \text{ Heads in 10 flips}\} = \binom{10}{6} p^6(1-p)^4 \propto p^6(1-p)^4.$$

The derivative of the likelihood function $L(p)$ is positively proportional to

$$\frac{\partial(p^6(1-p)^4)}{dp} = 6p^5(1-p)^4 - 4p^6(1-p)^3. \tag{1}$$

Setting the R.H.S. to zero and solving for p gives $p = \frac{3}{5}$. It is easy to see that the derivative is positive for $p < \frac{3}{5}$ and negative for $p > \frac{3}{5}$. Hence, $\hat{p}_{ML} = \frac{3}{5}$.

(b) Suppose it is known that $p = 0.05$. The biased coin is now tossed an unknown number n times during which 6 Heads are observed. What is the maximum likelihood estimate $\hat{n}_{ML}$ of n given this observation?

Solution: We know that X is Binomially distributed with parameters n and $p = 0.05$, and we observe $X = 6$. Thus, the likelihood of observing $X = 6$ is zero if $n < 6$. The likelihood function $L(n)$ for $n \geq 7$ is given by:

$$L(n) = P\{X = 6\} = \binom{n}{6}(0.05)^6(0.95)^{n-6}. \tag{2}$$

Because n must be an integer, we calculate the maximum without using derivatives. Instead, we calculate the ratio of $L(n-1), L(n)$ and $L(n+1)$ to determine the value of n that maximizes the likelihood $L(n)$. Note that

$$\begin{aligned}
1 &< \frac{L(n)}{L(n+1)} \\
&= \frac{\binom{n}{6}(0.05)^6(0.95)^{n-6}}{\binom{n+1}{6}(0.05)^6(0.95)^{n-5}} \\
&= \frac{\binom{n}{6}}{\binom{n+1}{6}(0.95)} \\
&= \frac{\frac{n!}{6!(n-6)!}}{\frac{(n+1)n!}{6!(n-5)(n-6)!}(0.95)} \\
&= \frac{1}{\frac{n+1}{n-5}(0.95)} \\
\implies & (n-5) > (n+1)(0.95) \\
\implies & n > 119.
\end{aligned}$$

Using similar math,

$$\frac{L(n)}{L(n-1)} > 1 \implies 0.95n > n - 6 \implies n < 120.$$

This implies that $L(n)$ strictly decreases for $n \geq 120$ and that $L(n)$ strictly increases for $n \leq 119$. Next, we check the values of $L(119)$ and $L(120)$ and find that $L(119)/L(120) = 1$. Thus, $\hat{n}_{ML} = 119$ or 120 and both values will be accepted as correct answers.

(c) The biased coin is tossed 9 times before the first Head is observed, i.e., the first Head is observed in the tenth flip. What is the maximum likelihood estimate $\hat{p}_{ML}$ of p given this observation?

Solution: If X is geometrically distributed random variable with parameters p, we observe $X = 10$. The likelihood function $L(p)$ for this observation is given by:

$$L(p) = (1-p)^9 p. \tag{3}$$

Taking the derivative of the likelihood function $L(p)$,

$$\frac{\partial((1-p)^9 p)}{dp} = -9(1-p)^8 p + (1-p)^9. \tag{4}$$

Setting the R.H.S. to zero and solving for p gives $p = \frac{1}{10}$. It is easy to see that the derivative is positive for $p < \frac{1}{10}$ and negative for $p > \frac{1}{10}$. Hence, $\hat{p}_{ML} = \frac{1}{10}$.

4. **[Life Insurance]**
An insurance company sold life insurance policies to 100,000 people for a premium of \$500 each. Assume that the probability of death of each insured person during the contract term is 0.001 and the deaths occur independently. In case of death, the company pays \$200,000 to a designated beneficiary. Let p_L denote the probability that the company loses money for the contract term.

(a) Find the upper bound on p_L using Markov's inequality.

Solution: The company gets $100,000 \cdot \$500 = \$5 \cdot 10^7$ in revenue from premiums. If there are more than $100,000 \cdot 500/200,000 = 250$ deaths among the insured people, the company will lose money. The death of any insured person follows a Bernoulli distribution with parameter $p = 0.001$. The total number of deaths follows a Binomial distribution. Let $X \sim \textit{Binomial}(n, p)$ be the number of insured people who die. From Markov's inequality,

$$P(X \geq 251) \leq \frac{E[X]}{251} = \frac{np}{251} = \frac{100000 \cdot 0.001}{251} = 0.398$$

Therefore,

$$p_L = P(X \geq 251) \leq 0.398$$

(b) Find the upper bound on p_L using Chebychev's inequality.

Solution: Using Chebychev's inequality,

$$\begin{aligned} P(|X - E[X]| \geq d) \leq P(|X - np| \geq d) &\leq \frac{np(1-p)}{d^2} \\ P(|X - 100| \geq 151) &\leq \frac{99.9}{151^2} \\ P(X \geq 251 \text{ or } X \leq -51) &\leq 0.00434 \\ P(X \geq 251) &\leq 0.00434 \end{aligned}$$

Therefore,

$$p_L = P(X \geq 251) \leq 0.00434$$

5. **[Election polls]**
You want to run a poll to estimate what fraction p of the population will vote for candidate A in the next election. Suppose you interview n people, and out of those, X say they will vote for candidate A.

(a) If you want to estimate p by $\hat{p} = \frac{X}{n}$ to within 0.01 with a confidence level of 96%, how large does n have to be? Justify your answer.

Solution: We know that for any $a > 0$,

$$P\left\{p \in \left(\hat{p} - \frac{a}{2\sqrt{n}}, \hat{p} + \frac{a}{2\sqrt{n}}\right)\right\} \geq 1 - \frac{1}{a^2}.$$

To obtain a 96% confidence,

$$0.96 = 1 - \frac{1}{a^2} \implies a = 5.$$

Now, in order to estimate p to within 0.01 of $\hat{p}_{ML}$, we need

$$\frac{a}{2\sqrt{n}} \leq 0.01 \implies \sqrt{n} \geq 50 \times a \implies n \geq 62,500.$$

(b) If $n = 2000$ people are actually polled, and out of those, 52% indicated they will vote for candidate A, what is the confidence interval corresponding to a confidence level of 96%?

Solution: Again, $a = 5$. Therefore the confidence interval is

$$\left(0.52 - \frac{5}{40\sqrt{5}}, 0.52 + \frac{5}{40\sqrt{5}}\right).$$

Or approximately:

$$(0.464, 0.576)$$