University of Illinois **Spring 2025**

ECE 313: Problem Set 3

Due: Friday, February 21 at 07:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.4 - 2.7.

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. [**Elvis Had a Twin Brother**]
   Elvis Presley had a twin brother who died at birth. This brother was either fraternal or identical. Using the fact that $\frac{1}{125}$ births are fraternal twins and $\frac{1}{300}$ are identical twins, find the probability that Elvis and his brother were identical twins, given that they were twins. Assume that boys and girls are equally likely for all births.

2. [**Bernoulli RVs and Conditional Probability**]
   Let Z_1, Z_2, Z_3 be independent Bernoulli random variables with parameters $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{2}$, respectively. Define a random variable $S = Z_1 + Z_2$ if $Z_3 = 1$, and $S = 2Z_1 - Z_2$ if $Z_3 = 0$.

   (a) Find $P(S = 1)$.

   (b) Find $P(Z_3 = 1|S = 1)$.

3. [**Mathematicians and Matchbooks**]
   A pipe-smoking mathematician carries, at all times, 2 matchboxes, one in his left and another in his right pocket. Each time he needs a match to light the pipe he is equally likely to reach out to either of the two pockets. Both matchboxes start with N matches. Consider the moment when the mathematician discovers that one of his matchboxes is empty. What is the probability that there are exactly k matches in the other box, where $k = 0, 1, \ldots, N$?

4. [**First Success**]
   An experiment is conducted until it results in success: the first step has probability $\frac{1}{2}$ to be successful, the second step (only conducted if the first step was unsuccessful) has probability $\frac{1}{3}$ to be successful, the third step (only conducted if the first two steps were unsuccessful) has probability $\frac{1}{4}$ to be successful. If none of the steps were successful, we repeat the experiment until success is achieved. Assuming the first step has cost 2, and the second and third steps have cost 1, what is the expectation of the cost until success?

5. **[Geometric RVs]**
   Suppose your cousin owns a lightbulb manufacturing company and determines that 3 out of every 75 bulbs are defective in average. Your cousin now repeatedly conducts such a trial: select a lightbulb randomly from a great many lightbulbs and test if it is defective.

   (a) What is the probability that your cousin will find the first defective lightbulb on the 6th trial?

   (b) Now, what if your cousin wants to know the probability (likelihood) that it takes at least 6 trials until he/she/they find the first defective lightbulb?

   (c) Determine the expected number of trials your cousin has to conduct until he/she/they find the first defective, as well as the standard deviation.

6. **[Bonus Problem: Communication Systems and Binomial Distributions]**
   A communication system consists of n components, each of which independently functions with probability p. The total system is functional if at least half of the components are functional. For what values of p is a $(2k+1)$-component system better than a $(2k-1)$-component system? Your answer should be a number or numerical interval for p.