University of Illinois **Spring 2025**

ECE 313: Problem Set 1: Problems and Solutions

Due: Friday, February 7 at 07:00:00 p.m.
Reading: *ECE 313 Course Notes,* Chapter 1 & Sections 2.1 - 2.2

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.
Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. [**Using set theory to calculate probabilities of events**]
Suppose A and B are two events defined on a probability space with $P(A) = 3/4$ and $P(B) = 1/2$.

   (a) If $B \subset A$, calculate $P(AB)$.
   Solution: If $B \subset A$, then $AB = B$ and $P(AB) = P(B) = 1/2$.

   (b) If $A \cup B = \Omega$, calculate $P(AB)$.
   Solution: Using relationship:

$$1 = P(\Omega) = P(A \cup B) = P(A) + P(B) - P(AB) = \frac{3}{4} + \frac{1}{2} - P(AB).$$

   Hence we get $P(AB) = 5/4 - 1 = 1/4$.

2. [**Fishing**]
Harry's backyard pond contains 6 goldfish and 4 catfish. All fish are equally likely to be caught.

   (a) Suppose that Harry catches a total of 5 fish (no fish are thrown back). Let G be the event that Harry catches exactly 3 goldfish. What is the probability of G i.e. $P(G)$?
   Solution: There are $\binom{10}{5}$ ways to catch 5 fish. Of these, there are $\binom{6}{3} \cdot \binom{4}{2}$ ways of catching 3 goldfish and 2 catfish. Thus,

$$\begin{aligned} P(G) &= \frac{\binom{6}{3} \cdot \binom{4}{2}}{\binom{10}{5}} \\ &= \frac{10}{21} \approx 0.476. \end{aligned}$$

(b) Assume event G occurs in the first catch and suppose that all 5 fish are returned to the pond. Harry starts fishing again. This time he catches a total of 3 fish. Let A be the event that among the caught set of 3 fish, exactly 2 goldfish are included that were also caught in the first catch? (Assume that fish do not learn from experience.) Find $P(A)$.
Solution: There are $\binom{10}{3}$ ways to catch 3 fish. Of these, there are $\binom{3}{2}$ ways of catching exactly two goldfish that were previously caught. Notice that there are $\binom{7}{1}$ ways of catching the third fish since the third fish should not be the remaining one goldfish caught in the first place. This gives:

$$\begin{aligned} P(A) &= \frac{\binom{3}{2} \cdot \binom{7}{1}}{\binom{10}{3}} \\ &= \frac{7}{40} = 0.175. \end{aligned}$$

3. **[Matching cards to boxes]**
Three boxes are placed on a table, with the i-th box containing a card with the number i, for $i = 1, 2, 3$. The cards are then removed from the boxes, randomly shuffled and placed back at random into the three boxes; all possibilities of which card is placed in which box are equally likely. Let X denote the number of boxes that get back their original card.

(a) Describe a suitable sample space Ω to describe the experiment. How many elements does Ω have?
Solution: Here are two (among many) possibilities:

$$\Omega = \{y_1y_2y_3 : y_1y_2y_3 \text{ is a permutation of } 123\}, \text{ or}$$

$$\Omega = \{y_1y_2y_3 : y_1, y_2, y_3 \in \{1, 2, 3\},\ y_1, y_2, y_3 \text{ distinct}\},$$

In both cases, y_i represents the number on the card placed back in box i, for $i = 1, 2, 3$. For example, 312 indicates that box 1 gets number 3, box 2 gets number 1, and box 3 gets number 2. There are $3! = 6$ elements in Ω, i.e., $|\Omega| = 6$.

(b) Find the pmf of X. Hint: First argue that X can only take the values 0, 1, and 3.
Solution: The possible values of X are 0,1, and 3; notice that X cannot take the value 2 since if two of the boxes get back their numbers, so will the third one.
There is only one outcome that contributes to $\{X = 3\}$, which is 123, so

$$p_X(3) = \frac{1}{6}.$$

The outcomes in $\{X = 1\}$ correspond to only one box getting back its number. For each such box i, there is only one outcome such that the other two boxes do not have their numbers. Therefore

$$\{X = 1\} = \{132, 321, 213\} \implies p_X(1) = \frac{3}{6} = \frac{1}{2}.$$

Now we can use the fact that the elements of the pmf sum up to one to conclude that

$$p_X(0) = 1 - \frac{1}{6} - \frac{1}{2} = \frac{1}{3}.$$

Alternatively, the outcomes in $\{X = 0\}$ correspond to no box getting back its number. Therefore there are 2 choices for y_1, i.e., 2 and 3, and for each of those choices, there is only one choice for numbers for the remaining two boxes. Therefore, $\{X = 0\} = \{231, 312\}$, and $p_X(0) = 1/3$.

(c) Find $E[X]$.
Solution: $E[X] = p_X(1) + 3p_X(3) = 1/2 + 1/2 = 1$.

(d) Find $\text{Var}(X)$.
Solution: By the definition of variance and part (c),

$$\text{Var}(X) = E[(X-1)^2] = 1^2 p_X(0) + 2^2 p_X(3) = \frac{1}{3} + \frac{4}{6} = 1.$$

Alternatively, we could first find $E[X^2] = 1^2 p_X(1) + 3^2 p_X(3) = 1/2 + 9/6 = 2$, and then $\text{Var}(X) = E[X^2] - E[X]^2 = 2 - 1 = 1$.

4. **[Gambling with Dice]**
You roll a fair die. If you roll an even number i, you win i dollars. If you roll an odd number, you lose m dollars. Let X denote the amount of money you win (a negative amount indicates a loss).

(a) Find the pmf of the random variable X. Note that the pmf will depend on m.
Solution: Since the die is fair,

$$p_X(2) = p_X(4) = p_X(6) = 1/6,$$
$$p_X(-m) = 1/2.$$

(b) If $E[X] = 0$, find m.
Solution: From the pmf we found in part (a),

$$E[X] = \frac{2}{6} + \frac{4}{6} + \frac{6}{6} - \frac{m}{2} = 2 - \frac{m}{2}.$$

Setting

$$2 - \frac{m}{2} = 0 \quad \Longrightarrow \quad m = 4.$$

(c) Fixing the value of m to the value you found in part (b), find $\text{Var}(X)$.
Solution:

$$\text{Var}(X) = E[X^2] - (E[X])^2 = 6^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + (-4)^2 \times \frac{1}{2} = \frac{52}{3},$$

since $E[X] = 0$ from part (b).

5. **[Properties of Expectation and Variance]**
Let X be a discrete-type random variable that takes the values $\{-2, -1, 0, 1, 2\}$ with pmf

$$p_X(-2) = \frac{1}{16},\ p_X(-1) = \frac{1}{8},\ p_X(0) = \frac{1}{2},\ p_X(1) = \frac{1}{4},\ p_X(2) = \frac{1}{16}.$$

(a) Compute the standard deviation of X.
Solution:

$$\sigma_X = \sqrt{Var(X)}, \quad Var(X) = E[X^2] - (E[X])^2.$$

By the formula of expectation

$$E[X] = -2\frac{1}{16} - \frac{1}{8} + \frac{1}{4} + 2\frac{1}{16} = \frac{1}{8}.$$

By LOTUS

$$E[X^2] = 4\frac{1}{16} + \frac{1}{8} + \frac{1}{4} + 4\frac{1}{16} = \frac{7}{8}.$$

Therefore, $Var(X) = \frac{55}{64} \approx 0.86$ and $\sigma_X = \sqrt{Var(X)} = \frac{\sqrt{55}}{8} \approx 0.93$.

(b) Compute $E[3X+1]$ and $E[3X^2+3X+1]$.
Solution: By linearity of expectation

$$E[3X+1] = 3E[X] + 1 = \frac{11}{8},$$

$$E[3X^2+3X+1] = 3E[X^2] + 3E[X] + 1 = \frac{32}{8} = 4.$$

(c) Compute $Var(3X+1)$ and $Var(3X^2+3X+1)$.
Solution: There is no linearity of variance, but we know that

$$Var(3X+1) = Var(3X) = 9Var(X) = \frac{495}{64} \approx 7.73$$

$$Var(3X^2+3X+1) = Var(3X^2+3X) = 9Var(X^2+X)$$

For the computation of $Var(X^2+X)$, we have

$$\begin{aligned} Var(X^2+X) &= E[(X^2+X)^2] - (E[X^2+X])^2 \\ &= E[X^4+2X^3+X^2] - (E[X^2]+E[X])^2 \\ &= E[X^4] + 2E[X^3] + E[X^2] - 1 \\ &= E[X^4] + 2E[X^3] - \frac{1}{8} \end{aligned}$$

and by LOTUS we have

$$E[X^3] = -8\frac{1}{16} - \frac{1}{8} + \frac{1}{4} + 8\frac{1}{16} = \frac{1}{8},$$

$$E[X^4] = 16\frac{1}{16} + \frac{1}{8} + \frac{1}{4} + 16\frac{1}{16} = \frac{19}{8}.$$

Therefore,

$$Var(X^2+X) = \frac{19}{8} + \frac{2}{8} - \frac{1}{8} = \frac{5}{2}$$

and thus

$$Var(3X^2+3X+1) = 9 \cdot \frac{5}{2} = \frac{45}{2}.$$