University of Illinois **Spring 2024**

ECE 313: Exam I

Monday, February 26, 2024
7:00 p.m. — 8:15 p.m.

Name: **(in BLOCK CAPITALS)** ____

NetID: ____

Signature: ____

Section: ☐ A, MWF at 10 am (Milenkovic) ☐ B, MWF at 11 am (Katselis) ☐ C, MWF at 1 pm (Shanbhag) ☐ D, MWF at 2pm (Bastopcu) ☐ CSP, Chicago (Shanbhag)

Instructions

This exam is closed book and closed notes except for one 8.5”×11” sheet of notes: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, microscopes, etc. are not allowed.

The exam consists of four problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. **Draw a small box around each of your final numerical answers.**

Grading

1. 20 points	________
2. 35 points	________
3. 30 points	________
4. 15 points	________
Total (100 points)	________

1. **[20 points]** Suppose that a fair coin is independently tossed 3 times. Let A_{ij} be the event that the ith and jth tosses produce the same outcome. Consider the events A_{12}, A_{13}, A_{23}.

   (a) [10 points] Are these events pairwise independent? Justify your answer.

   (b) [10 points] Are they independent? Justify your answer.

2. **[35 points]** Suppose that the random variable X has the following pmf:

$$p_X(k) = \begin{cases} c_1 \left(\frac{1}{2}\right)^k, & \text{if } k = -2, -1, 0, \\ c_2, & \text{if } k = m, \\ 0, & \text{otherwise.} \end{cases}$$

Note: If you cannot find a solution for some parts of the problem, use the symbols without knowing their numerical values to express the solutions for other parts.

(a) [10 points] Given that $p_X(m) = 5p_X(0)$, what are the values of the constants c_1 and c_2 so that we have a valid pmf?

(b) [10 points] Find the value of m such that $E[X] = 0$, using the constants c_1, c_2 found in part (a).

(c) [5 points] With the constants c_1, c_2, found in part (a), and the value of m found in part (b), find the variance of the random variable X, i.e., $Var(X)$.

(d) [10 points] Find $P(|X| \geq 2)$ and $E[|X|]$, for the values of the constants c_1, c_2 found in part (a) and the value of m found in part (b).

3. **[30 points]** Suppose that you play the lottery by purchasing a ticket every week. The probability of winning a \$5 prize is 10^{-2}, while the probability of winning the main prize is 10^{-6}, with each weekly draw performed independently of others. Assume that the year has exactly 52 weeks.

   (a) [10 points] What is the expected number of times you win the \$5 prize? What is the probability that you win the main prize exactly once in the year? You can provide expressions for your solutions without computing the actual numerical values.

   (b) [10 points] Find the expected number of tickets that you need to purchase until you win the main prize (suppose you have enough money to purchase weekly tickets "forever"; also, once you get the main prize, there is no reason for you to ever gamble again).

(c) [10 points] Find the expected number of tickets that you need to purchase until you win two \$5 awards in consecutive order (one after another, and for the first time).

4. **[15 points]** A random variable X takes values in the set $\mathcal{S} = \{1, 3, 5, \ldots, 2n+1\}$ with equal probability.

   (a) [10 points] If $X = 13$ is observed, find the maximum likelihood estimate n_{ML} of n.

   (b) [5 points] Suppose three independent observations $X_1 = 13$, $X_2 = 7$ and $X_3 = 9$ are made. Propose a reasonable estimate for n based on these three observations.