University of Illinois Spring 2024

ECE 313: Exam II

Monday, April 8, 2024
7:00 p.m. — 8:15 p.m.

1. (a) Let X denote the number of defective light bulbs out of the 3 samples. Let H_0 denote the hypothesis that the company's claim is true, and H_1 denote the hypothesis that the quality assurance team's hypothesis is true. If H_0 is true, $X \sim \text{Bi}(3, \frac{1}{4})$; otherwise, if H_1 is true, $X \sim \text{Bi}(3, \frac{1}{2})$. The likelihood matrix is then:

	$X = 0$	$X = 1$	$X = 2$	$X = 3$
H_0	27/64	27/64	9/64	1/64
H_1	1/8	3/8	3/8	1/8

The ML decision rule is thus to declare H_0 if $X = 0, 1$ and declare H_1 if $X = 2, 3$.

(b) According to your definition of H_0 and H_1, your $p_{false\,alarm}$ may equal the value of p_{miss} in this solution, and vice versa. As long as your definitions are consistent, however, your p_e should be the same.

$$p_{false\,alarm} = P(declare\,H_1|H_0) \quad (1)$$
$$= P(X = 2, 3|H_0) \quad (2)$$
$$= \frac{5}{32} \quad (3)$$

$$p_{miss} = P(declare\,H_0|H_1) \quad (4)$$
$$= P(X = 0, 1|H_1) \quad (5)$$
$$= \frac{1}{2} \quad (6)$$

$$p_e = p_{false\,alarm} \cdot \frac{1}{3} + p_{miss} \cdot \frac{2}{3} \quad (7)$$
$$= \frac{5}{96} + \frac{1}{3} \quad (8)$$
$$= \frac{37}{96} \quad (9)$$

(c) The MAP joint probability matrix is:

	$X=0$	$X=1$	$X=2$	$X=3$
H_0	$9/64$	$9/64$	$3/64$	$1/192$
H_1	$1/12$	$1/4$	$1/4$	$1/12$

Therefore H_0 is chosen when $X=0$ and H_1 is chosen $X=1,2,3$

(d) Remember that our false alarm and miss probabilities are conditional. Refer to the ML table for the respective values.

$$p_{false\,alarm} = P(H_1\,declared|H_0) \tag{10}$$
$$= P(X=1,2,3|H_0) \tag{11}$$
$$= \frac{37}{64} \tag{12}$$

$$p_{miss} = P(H_0\,declared|H_1) \tag{13}$$
$$= P(X=0|H_1) \tag{14}$$
$$= \frac{1}{8} \tag{15}$$

$$p_e = p_{miss} \cdot \frac{2}{3} + p_{false\,alarm} \cdot \frac{1}{3} \tag{16}$$
$$= \frac{53}{192} \tag{17}$$

Note that $\frac{53}{192} < \frac{37}{96}$.

2. (a)

$$P(N_5 - N_3 = 0) = \frac{e^{-2\lambda}(2\lambda)^0}{0!}$$
$$= e^{-10}$$

Alternatively, you can let $T \sim exp(\lambda)$ denote the time of first arrival after $t = 3$, and thus

$$\begin{aligned} P(N_5 - N_3 = 0) &= P(T > 2) \\ &= 1 - P(T \leq 2) \\ &= e^{-2\lambda} = e^{-10} \end{aligned}$$

(b)

$$\begin{aligned} &P(N_2 - N_1 = 2 \cap N_5 - N_1 = 5) \\ &= P(N_2 - N_1 = 2 \cap N_5 - N_2 = 3) \\ &= P(N_2 - N_1 = 2)P(N_5 - N_2 = 3) \\ &= \frac{e^{-\lambda}\lambda^2}{2!}\frac{e^{-3\lambda}(3\lambda)^3}{3!} \\ &= \frac{28125}{4}e^{-20} \end{aligned}$$

(c)

$$\begin{aligned} P(N_5 \geq 3 | N_3 = 2) &= \frac{P(N_5 \geq 3 \cap N_3 = 2)}{P(N_3 = 2)} \\ &= \frac{P(N_5 - N_3 \geq 1 \cap N_3 = 2)}{P(N_3 = 2)} \\ &= P(N_5 - N_3 \geq 1) \\ &= 1 - P(N_5 - N_3 = 0) \\ &= 1 - e^{-10} \end{aligned}$$

Alternatively, you can let $T \sim exp(\lambda)$ denote the time of first arrival after $t = 3$, and thus

$$\begin{aligned} P(N_5 - N_3 \geq 1) &= P(T \geq 2) \\ &= 1 - e^{-10} \end{aligned}$$

3. (a) We have

$$(\text{number of times a number other than 6 shows}) = 100 - X.$$

Therefore

$$A = \{X \leq 180 - X + 4\} = \{X \leq 92\}.$$

Since a 6 (success) occurs with probability 1/6 on every die roll, $X \sim \text{Binom}(180, 1/6)$.

(b) Since $X \sim \text{Binom}(180, 1/6)$, therefore

$$P(A) = P\{X \leq 92\} = \sum_{k=0}^{92} \binom{180}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{180-k} = \left(\frac{5}{6}\right)^{180} \sum_{k=0}^{92} \binom{180}{k} 5^{-k}.$$

(c) Note that $E[X] = np = 30$, $\text{Var}(X) = (180)p(1-p) = 25$, and $\sigma = \sqrt{\text{Var}(X)} = 5$. Therefore

$$\begin{aligned} P(A) = P\{X \leq 92\} &\approx P\{\tilde{X} \leq 92.5\} \\ &= P\left\{\frac{\tilde{X} - 30}{5} \leq \frac{92.5 - 30}{5}\right\} \\ &= \Phi\left(\frac{92.5 - 30}{5}\right) = \Phi(12.5). \end{aligned}$$

where the approximation comes from Gaussian approximation with continuity correction.

(d) We have

$$\begin{aligned} P(B) = P\{X = 55\} = P\{54.5 < X < 55.5\} &\approx P\{54.5 < \tilde{X} < 55.5\} \\ &= P\left\{\frac{54.5 - 30}{5} < \frac{\tilde{X} - 30}{5} < \frac{55.5 - 30}{5}\right\} \\ &= Q\left(\frac{54.5 - 30}{5}\right) - Q\left(\frac{55.5 - 30}{5}\right) = Q(4.9) - Q(5.1). \end{aligned}$$

4. (a) Note that X is continuous and a uniform random variable.

$$P(1 < X < 2) = \frac{1}{3}(2-1) = \frac{1}{3}$$

$$P(1 \leq X \leq \frac{3}{2}) = \frac{1}{3}(\frac{3}{2}-1) = \frac{1}{6}$$

(b)

$$P(X \leq 3Y) = P(3X^2 - X \geq 0) = P(X(3X-1) \geq 0)$$

$X(3X-1)$ is an upward parabola with roots $X = 0, \frac{1}{3}$. Therefore, $X(3X-1) \geq 0$ for $X \leq 0$ and $X \geq \frac{1}{3}$.

$$\begin{aligned}
P(X \leq 3Y) &= P(3X^2 - X \geq 0) \\
&= P(X(3X-1) \geq 0) \\
&= P(X \leq 0 \cup X \geq \frac{1}{3}) \\
&= P(X \leq 0) + P(X \geq \frac{1}{3}) \\
&= \frac{1}{3}(3 - \frac{1}{3}) \\
&= \frac{8}{9}
\end{aligned}$$